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An Asymptotic Expansion Approach to Currency Options with a Market Model of Interest Rates under Stochastic Volatility Processes of Spot Exchange Rates

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Abstract

This paper proposes an asymptotic expansion scheme of currency options with a libor market model of interest rates and stochastic volatility models of spot exchange rates. In particular, we derive closed-form approximation formulas for the density functions of the underlying assets and for pricing currency options based on a third order asymptotic expansion scheme; we do not model a foreign exchange rate's variance such as in Heston[1993], but its volatility that follows a general time-inhomogeneous Markovian process. Further, the correlations among all the factors such as domestic and foreign interest rates, a spot foreign exchange rate and its volatility, are allowed. Finally, numerical examples are provided and the pricing formula are applied to the calibration of volatility surfaces in the JPY/USD option market.

1 Introduction

In this paper we propose new approximation formulas for the density functions of foreign exchange rates and for the valuation of currency options under stochastic volatility processes of spot exchange rates in stochastic interest rates environment. In particular, we use models of volatility processes, not variance processes such as in Heston[1993], and apply a libor market model developed by Brace, Gatarek and Musiela[1998] and Miltersen, Sandmann and Sondermann[1997] to modeling term structures of interest rates. Moreover, the correlations among all the factors such as domestic and foreign interest rates, a spot foreign exchange rate and its volatility, are allowed.

Currency options with maturities beyond one year become common in global currencies' markets and even smiles or skews for those maturities are frequently observed. Because it is well known that the effects of interest rates become more substantial in longer maturities, we have to take term structure models into account for the currency options. Further, stochastic volatility models of foreign exchange rates are necessary for calibration of smiles and skews. As for term structure models, market models become popular in matured interest rates markets since calibrations of caps, floors and swaptions are required and market models are regarded as most useful.

Hence, our objective is to develop a model with stochastic volatilities of exchange rates and with a libor market model of interest rates. Moreover, a closed-form formula is desirable in practice especially for calibrations since they are very time consuming by numerical methods such as Monte Carlo simulation. Because it is impossible to obtain an exact closed-form formula, we derive a closed-form approximation formula by an asymptotic expansion up to the third order where a volatility of a spot exchange rate follows a general time-inhomogeneous Markovian process, and domestic and foreign interest rates are generated by a libor market model.

Garman and Kohlhagen[1983] and Grabbe[1983] started research for currency options based on a contingent claim analysis; the framework of Black and Scholes[1973], Merton[1973] and Black[1976] was directly applied to pricing currency options. Grabbe[1983]'s formula also included the case of stochastic interest rates following Gaussian processes though he did not specify the processes explicitly. Rumsey[1991] and Melino and Turnbull[1991] developed models under the deterministic interest rates assumption.

Amin and Jarrow[1991] and Hilliard, Madura and Tucker[1991] derived formulas of currency options with Gaussian stochastic interest rates; in particular, Amin and Jarrow[1991] combined term structure models under the framework of Heath, Jarrow and Morton [1992](HJM[1992]) with currency options.

Amin and Bodurtha[1995] and Takahashi and Tokioka[1999] gave numerical solutions to price currency American options with stochastic interest rates by lattice methods; Amin and Bodurtha[1995] used HJM[1992] models and Takahashi and Tokioka[1999] applied Hull and White[1990,1994] term structure models. Dempster and Hutton[1997] considered terminable (Bermudan) differential swaps with Gaussian interest rates models by using the partial differential equations(PDE) approach.

Schlögl[2002] extended market models to a cross-currency framework. He did not take stochastic volatilities into account and focus on cross currency derivatives such as differential swaps and options on differential swaps as examples; currency options were not considered. Mikkelsen[2001] considered cross-currency options with market models of interest rates and deterministic volatilities of spot exchange rates by simulation. Piterbarg[2005] developed a model for cross-currency derivatives such as Power-Reverse-Dual-Currency(PRDC) swaps with calibration to currency options; neither market models nor stochastic volatility models were used.

Our asymptotic expansion approach have been applied to a broad class of Itô processes appearing in finance. It started with pricing average options; Kunitomo and Takahashi[1992] derived a first order approximation and Yoshida[1992b] applied an asymptotic expansion method developed in statistics for stochastic processes. Takahashi[1995,1999] presented second or third order schemes for pricing various options in a general Markovian setting with a constant interest rate. Kunitomo and Takahashi[2001] provided approximation formulas for pricing bond options and average options on interest rates in term structure models of HJM[1992] which is not necessarily Markovian.

Moreover, Takahashi and Yoshida[2004,2005] extended the method to dynamic portfolio problems in a general Markovian setting and proposed a new variance reduction scheme of Monte Carlo simulation with an asymptotic expansion. For mathematical validity of the method based on Watanabe[1987] in the Malliavin calculus, see Chapter 7 of Malliavin and Thalmaier[2006], Yoshida[1992a], Kunitomo and Takahashi[2003] and Takahashi and Yoshida[2004,2005].

Other applications and extensions of asymptotic expansions to numerical problems in finance are found as follows: Kawai[2003], Kobayashi,Takahashi and Tokioka[2003], Takahashi and Saito[2003], Lütkebohmert[2004a,b], Kunitomo and Takahashi[2004], Kunitomo and Kim[2005], Muroi[2005], Takahashi[2005], Matsuoka,Takahshi and Uchida[2006], Takahashi and Uchida[2006].

The organization of the paper is as follows: After the next section describes basic structure of our model, Section 3 derives approximation formulas. Section 4 shows numerical examples and the final section states conclusion. Appendix A gives the concrete expressions of coefficients in the asymptotic expansions, and Appendix B presents formulas used in Appendix A.

2 European Currency Options with a Market Model of Interest Rates and Stochastic Volatility Models of Spot Exchange Rates

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{0\leq t\leq T^*<\infty})$ be a complete probability space with filtration satisfying the usual conditions. First we briefly state the basics of European currency options. The payoffs of call and put options with maturity $T \in (0, T^*]$ and strike rate $K > 0$ are expressed as $(S(T) - K)^+$ and $(K - S(T))^+$ respectively where $S(t)$ denotes the spot exchange rate at time $t \geq 0$ and x^+ denotes $\max(x, 0)$. In this paper we will concentrate on the valuation of a call option since the value of a put option can be obtained through the put-call parity or similar method. We also note that the spot exchange rate $S(T)$ can be expressed in terms of a foreign exchange forward(forex forward) rate with the same maturity T. That is, $S(T) = F_T(T)$ where $F_T(t)$, $t \in [0, T]$ denotes the time t value of the forex forward rate with maturity T. It is well known that the arbitrage-free relation between the forex spot rate and the forex forward rate are given by $F_T(t) = S(t)\frac{P_f(t,T)}{P_d(t,T)}$ where $P_d(t, T)$ and $P_f(t, T)$ denote the time t values of domestic and foreign zero coupon bonds with maturity T respectively.

Hence, our objective is to obtain the present value of the payoff $(F_T(T) - K)^+$. In particular, we need to evaluate:

$$V(0;T,K) \quad = \quad P_d(0,T)\mathbf{E}\left[(F_T(T)-K)^+\right] \tag{1}$$

where $V(0;T,K)$ denotes the value of an European call option at time 0 with maturity T and strike rate K, and $\mathbf{E}[\cdot]$ denotes the expectation operator under EMM(Equivalent Martingale Measure) of numeraire of the domestic zero coupon bond maturing at T (we use a term of *the domestic terminal measure* in what follows). Then, the distribution of $F_T(T)$ under the domestic terminal measure is necessary for pricing the option. For this objective, a market model and stochastic volatility models are applied to modeling interest rates' and the spot exchange rate's dynamics respectively.

In the rest of this section, we describe briefly the model to which an asymptotic expansion approach will be applied in the following sections, where the appropriate regularity conditions are implicitly assumed without mentioned.

We first define domestic and foreign forward interest rates as $f_{dj}(t) = \left(\frac{P_d(t,T_j)}{P_d(t,T_{j+1})}-1\right)\frac{1}{\tau_j}$ and $f_{fj}(t) = \left(\frac{P_f(t,T_j)}{P_f(t,T_{j+1})}-1\right)\frac{1}{\tau_j}$ respectively, where $j = n(t), n(t)+1, \cdots, N$, $\tau_j = T_{j+1}-T_j$, and $P_d(t,T_j)$ and $P_f(t,T_j)$ denote the prices of domestic/foreign zero coupon bonds with maturity T_j at time $t(\leq T_j)$ respectively; $n(t) = \min\{i : t \leq T_i\}$. We also define spot interest rates to the nearest fixing date denoted by $f_{d,n(t)-1}(t)$ and $f_{f,n(t)-1}(t)$ as $f_{d,n(t)-1}(t) = \left(\frac{1}{P_d(t,T_{n(t)})}-1\right)\frac{1}{(T_{n(t)}-t)}$ and $f_{f,n(t)-1}(t) = \left(\frac{1}{P_f(t,T_{n(t)})}-1\right)\frac{1}{(T_{n(t)}-t)}$. Finally, we set $T = T_{N+1}$ and will abbreviate $F_{T_{N+1}}(t)$ to $F_{N+1}(t)$ in what follows.

$\mathbf{R}_{++}$-valued processes of domestic forward interest rates under the domestic terminal measure can be specified as; for $j = n(t)-1, n(t), n(t)+1, \cdots, N$,

$$f_{dj}(t) \quad = \quad f_{dj}(0) + \int_0^t \left\{ -f_{dj}(u)\tilde{\gamma}_{dj}^{'}(u)\sum_{i=j+1}^{N}\frac{\tau_i f_{di}(u)\tilde{\gamma}_{di}(u)}{1+\tau_i f_{di}(u)} \right\} du + \int_0^t f_{dj}(u)\tilde{\gamma}_{dj}^{'}(u)dW_u \tag{2}$$

where $x^{'}$ denotes the transpose of x, and W is a D dimensional Brownian motion under the domestic terminal measure; $\tilde{\gamma}_{dj}(u)$ is a function of time-parameter u. Similarly, $\mathbf{R}_{++}$-valued processes of foreign ones under the foreign terminal measure are specified as

$$f_{fj}(t) \quad = \quad f_{fj}(0) + \int_0^t \left\{ -f_{fj}(u)\tilde{\gamma}_{fj}^{'}(u)\sum_{i=j+1}^{N}\frac{\tau_i f_{fi}(u)\tilde{\gamma}_{fi}(u)}{1+\tau_i f_{fi}(u)} \right\} du + \int_0^t f_{fj}(u)\tilde{\gamma}_{fj}^{'}(u)dW_u^f \tag{3}$$

where W^f is a D dimensional Brownian motion under the foreign terminal measure and $\tilde{\gamma}_{fj}(u)$ is a function of u.

Finally, it is assumed that the spot exchange rate $S(t)$ and its volatility $\tilde{\sigma}(t)$ follow $\mathbf{R}_{++}$-valued stochastic processes below under the domestic risk neutral measure:

$$\begin{aligned} S(t) \quad &= \quad S(0) + \int_0^t S(u)(r_d(u)-r_f(u))du + \int_0^t S(u)\tilde{\sigma}(u)\bar{\sigma}^{'}d\hat{W}_u \\ \tilde{\sigma}(t) \quad &= \quad \tilde{\sigma}(0) + \int_0^t \hat{\mu}(\tilde{\sigma}(u),u)du + \int_0^t \tilde{\omega}^{'}(\tilde{\sigma}(u),u)d\hat{W}_u \end{aligned} \tag{4}$$

where $\hat{W}$ is a D dimensional Brownian motion under the domestic risk neutral measure and $r_d(u)$ and $r_f(u)$ denote domestic and foreign instantaneous spot interest rates respectively; $\bar{\sigma}$ denotes a D dimensional constant vector satisfying $||\bar{\sigma}|| = 1$, and $\tilde{\omega}(x,u)$ is a function of x and u. In the model, the volatility of a volatility process is allowed to be general time-inhomogeneous Markovian while the interest rates' volatilities are specified as a log-normal structure. Note that the correlations' structure among domestic/foreign interest rates, the spot exchange rate and its volatility can be represented by $\tilde{\gamma}_{dj}(t)$, $\tilde{\gamma}_{fj}(t)$, $\bar{\sigma}$ and $\tilde{\omega}(\tilde{\sigma}(t),t)$.

Moreover, we note the following well known relations among Brownian motions under different measures;

$$\begin{aligned} W_u \quad &= \quad \hat{W}_u - \int_0^u \tilde{\sigma}_{dN+1}(s)ds \\ &= \quad W_u^f + \int_0^u \{\tilde{\sigma}_{fN+1}(s) - \tilde{\sigma}_{dN+1}(s) + \tilde{\sigma}(s)\bar{\sigma}\}ds \end{aligned}$$

where $\tilde{\sigma}_{dN+1}(u)$ and $\tilde{\sigma}_{fN+1}(u)$ are volatilities of the domestic and foreign zero coupon bonds with the maturity T_{N+1}, that is,

$$\tilde{\sigma}_{dN+1}(u) := \sum_{i \in J_{N+1}(u)} \frac{-\tau_i f_{di}(u)\tilde{\gamma}_{di}(u)}{1+\tau_i f_{di}(u)}$$

$$\tilde{\sigma}_{fN+1}(u) := \sum_{i \in J_{N+1}(u)} \frac{-\tau_i f_{fi}(u)\tilde{\gamma}_{fi}(u)}{1+\tau_i f_{fi}(u)}$$

and $J_{j+1}(t) = \{n(t)-1, n(t), n(t)+1, \cdots, j\}$. Because $\gamma_{fj}(t) = 0$ and $\gamma_{dj}(t) = 0$ for all j such that $T_j \leq t$, the set of indices $J_{j+1}(t)$ can be changed into $\hat{J}_{j+1} := \{0, 1, \cdots, j\}$, which does not depend on t.

Using above equations, we can unify expressions of those processes under different measures into ones under the same measure, the domestic terminal measure:

$$\begin{aligned} f_{fj}(t) &= f_{fj}(0) + \int_0^t f_{fj}(u)\tilde{\gamma}_{fj}^{'}(u) \left\{ \sum_{i \in \hat{J}_{j+1}} \frac{\tau_i f_{fi}(u)\tilde{\gamma}_{fi}(u)}{1+\tau_i f_{fi}(u)} - \sum_{i \in \hat{J}_{N+1}} \frac{\tau_i f_{di}(u)\tilde{\gamma}_{di}(u)}{1+\tau_i f_{di}(u)} \right\} du \\ &\quad - \int_0^t f_{fj}(u)\tilde{\gamma}_{fj}^{'}(u)\tilde{\sigma}(u)\bar{\sigma} du + \int_0^t f_{fj}(u)\tilde{\gamma}_{fj}^{'}(u) dW_u \quad (5) \\ \tilde{\sigma}(t) &= \tilde{\sigma}(0) + \int_0^t \mu(u) du + \int_0^t \tilde{\omega}^{'}(\tilde{\sigma}(u), u) dW_u \quad (6) \end{aligned}$$

where $\mu(u)$ is defined as

$$\mu(u) := \hat{\mu}(\tilde{\sigma}(u), u) + \tilde{\omega}^{'}(\tilde{\sigma}(u), u)\tilde{\sigma}_{dN+1}(u).$$

Next, we consider the process of the forex forward $F_{N+1}(t)$. Since $F_{N+1}(t)$ can be expressed as

$$F_{N+1}(t) := S(t)\frac{P_f(t, T_{N+1})}{P_d(t, T_{N+1})}, \quad (7)$$

we easily notice that it is a martingale under the domestic terminal measure, and we can obtain its process under that measure by applying Itô's formula to (7):

$$\begin{aligned} F_{N+1}(t) &= F_{N+1}(0) + \int_0^t [\tilde{\sigma}_{fN+1}(u) - \tilde{\sigma}_{dN+1}(u) + \tilde{\sigma}(u)\bar{\sigma}]^{'} F_{N+1}(u) dW_u \\ &= F_{N+1}(0) \\ &\quad + \int_0^t \left[\sum_{j \in \hat{J}_{N+1}} \frac{-\tau_j f_{fj}(u)\tilde{\gamma}_{fj}(u)}{1+\tau_j f_{fj}(u)} - \sum_{j \in \hat{J}_{N+1}} \frac{-\tau_j f_{dj}(u)\tilde{\gamma}_{dj}(u)}{1+\tau_i f_{dj}(u)} + \tilde{\sigma}(u)\bar{\sigma} \right]^{'} F_{N+1}(u) dW_u. \quad (8) \end{aligned}$$

3 An Approximation Scheme based on an Asymptotic Expansion Approach

An asymptotic expansion approach describes the processes of forward rates and that of a foreign exchange rate's volatility as $f_{dj}^{(\epsilon)}(t)$, $f_{fj}^{(\epsilon)}(t)$ and $\sigma^{(\epsilon)}(t)$ respectively, all of which explicitly depend upon a parameter $\epsilon \in (0, 1]$, and expands the processes around $\epsilon = 0$, that is asymptotic expansions are made around deterministic processes.

First, the processes of $f_{dj}^{(\epsilon)}(t)$, $f_{fj}^{(\epsilon)}(t)$ and $\sigma^{(\epsilon)}(t)$ are redefined as follows; for $j = n(t)-1, n(t), n(t)+1, \cdots, N$,

$$\begin{aligned} f_{dj}^{(\epsilon)}(t) &= f_{dj}(0) + \epsilon^2 \int_0^t \left\{ -f_{dj}^{(\epsilon)}(u)\gamma_{dj}^{'}(u) \sum_{i=j+1}^N \frac{\tau_i f_{di}^{(\epsilon)}(u)\gamma_{di}(u)}{1+\tau_i f_{di}^{(\epsilon)}(u)} \right\} du + \epsilon \int_0^t f_{dj}^{(\epsilon)}(u)\gamma_{dj}^{'}(u) dW_u \quad (9) \\ f_{fj}^{(\epsilon)}(t) &= f_{fj}(0) + \epsilon^2 \int_0^t \left\{ -f_{fj}^{(\epsilon)}(u)\gamma_{fj}^{'}(u) \sum_{i=j+1}^N \frac{\tau_i f_{fi}^{(\epsilon)}(u)\gamma_{fi}(u)}{1+\tau_i f_{fi}^{(\epsilon)}(u)} \right\} du + \epsilon \int_0^t f_{fj}^{(\epsilon)}(u)\gamma_{fj}^{'}(u) dW_u^f \quad (10) \\ \sigma^{(\epsilon)}(t) &= \sigma(0) + \int_0^t \hat{\mu}(\sigma^{(\epsilon)}(u), u, \epsilon) du + \epsilon \int_0^t \omega^{'}(\sigma^{(\epsilon)}(u), u) d\hat{W}_u \quad (11) \end{aligned}$$

where $\tilde{\gamma}_{dj}(t)$, $\tilde{\gamma}_{fj}(t)$, $\tilde{\sigma}(t)$ and $\tilde{\omega}(\sigma^{(\epsilon)}(t),t)$ in the previous section are replaced by $\epsilon\gamma_{dj}(t)$, $\epsilon\gamma_{fj}(t)$, $\epsilon\sigma(t)$, and $\epsilon\omega(\sigma^{(\epsilon)}(t),t)$ respectively.

Hence, the processes of $f_{fj}^{(\epsilon)}(t)$, $\sigma^{(\epsilon)}(t)$ and $F_{N+1}^{(\epsilon)}(t)$ under the domestic terminal measure are expressed as follows:

$$\begin{aligned}
f_{fj}^{(\epsilon)}(t) &= f_{fj}^{(\epsilon)}(0)+\epsilon^2\int_0^t f_{fj}^{(\epsilon)}(u)\gamma_{fj}^{'}(u)\left\{\sum_{i\in\hat{J}_{j+1}}\frac{\tau_i f_{fi}^{(\epsilon)}(u)\gamma_{fi}(u)}{1+\tau_i f_{fi}^{(\epsilon)}(u)}-\sum_{i\in\hat{J}_{N+1}}\frac{\tau_i f_{di}^{(\epsilon)}(u)\gamma_{di}(u)}{1+\tau_i f_{di}^{(\epsilon)}(u)}\right\}du \\
&\quad -\epsilon^2\int_0^t f_{fj}^{(\epsilon)}(u)\gamma_{fj}^{'}(u)\sigma^{(\epsilon)}(u)\bar{\sigma}du+\epsilon\int_0^t f_{fj}^{(\epsilon)}(u)\gamma_{fj}^{'}(u)dW_u \qquad (12)\\
\sigma^{(\epsilon)}(t) &= \sigma(0)+\int_0^t\mu^{(\epsilon)}(u)du+\epsilon\int_0^t\omega^{'}(\sigma^{(\epsilon)}(u),u)dW_u \qquad (13)\\
F_{N+1}^{(\epsilon)}(t) &= F_{N+1}(0)+\epsilon\int_0^t[\sigma_{fN+1}^{\epsilon}(u)-\sigma_{dN+1}^{\epsilon}(u)+\sigma^{(\epsilon)}(u)\bar{\sigma}]^{'}F_{N+1}^{(\epsilon)}(u)dW_u\\
&= F_{N+1}(0)\\
&\quad +\epsilon\int_0^t\left[\sum_{j\in\hat{J}_{N+1}}\frac{-\tau_j f_{fj}^{(\epsilon)}(u)\gamma_{fj}(u)}{1+\tau_j f_{fj}^{(\epsilon)}(u)}-\sum_{j\in\hat{J}_{N+1}}\frac{-\tau_j f_{dj}^{(\epsilon)}(u)\gamma_{dj}(u)}{1+\tau_i f_{dj}^{(\epsilon)}(u)}+\sigma^{(\epsilon)}(u)\bar{\sigma}\right]^{'}F_{N+1}^{(\epsilon)}(u)dW_u \qquad (14)
\end{aligned}$$

Under certain appropriate conditions on $\mu^{(\epsilon)}(t)$ and $\omega(\sigma^{(\epsilon)}(t),t)$, the system of SDEs (9), (12), (13) and (14) have their unique solutions $f_{dj}^{(\epsilon)}(t)$, $f_{fj}^{(\epsilon)}(t)$, $\sigma^{(\epsilon)}(t)$ and $F_{N+1}^{(\epsilon)}(t)$.

Next, we expand forward rates' and volatility's processes up to the second order of ϵ(ϵ^2-order) around $\epsilon=0$ to obtain the third order asymptotic expansion of forex forward rate $F_{N+1}^{(\epsilon)}(t)$. These expansions can be obtained by differentiating the right hand sides of the equations (9), (12), (13) and (14) with respect to ϵ at $\epsilon=0$. The result is stated as the following lemma after some notes.

We notice that the mappings $\epsilon\rightarrow f_{dj}^{(\epsilon)}(t)$, $\epsilon\rightarrow f_{fj}^{(\epsilon)}(t)$, $\epsilon\rightarrow\sigma^{(\epsilon)}(t)$ and $\epsilon\rightarrow F_{N+1}^{(\epsilon)}(t)$ are all smooth under the additional assumptions that $\mu^{(\epsilon)}(t)$ and $\omega(\sigma^{(\epsilon)}(t),t)$ are smooth and that their derivatives of any order are bounded. Moreover, the validity of asymptotic expansions appearing in what follows is justified by Watanabe Theory(Watanabe[1987]) in Malliavin calculus. However, since this paper concentrates on the practical applications of an asymptotic expansion, the details are omitted (See Bichteler, Gravereaux and Jacod[1987], Yoshida[1992a,1992b], Lütkebohmert[2004a,b], Kunitomo and Takahashi[2003] or Takahashi and Yoshida[2005]).

Lemma 1 *The asymptotic expansions of domestic/foreign forward rates and the spot exchange rate's volatility are given as follows:*

$$\begin{aligned}
f_{dj}^{(\epsilon)}(t) &= f_{dj}(0)+\epsilon A_{dj}^{(1)}(t)+\epsilon^2A_{dj}^{(2)}(t)+o(\epsilon^2) \qquad (15)\\
f_{fj}^{(\epsilon)}(t) &= f_{fj}(0)+\epsilon A_{fj}^{(1)}(t)+\epsilon^2A_{fj}^{(2)}(t)+o(\epsilon^2) \qquad (16)\\
\sigma^{(\epsilon)}(t) &= \sigma(t)+\epsilon A_{\sigma}^{(1)}(t)+\epsilon^2A_{\sigma}^{(2)}(t)+o(\epsilon^2) \qquad (17)
\end{aligned}$$

where

$$\begin{aligned}
A_{dj}^{(1)}(t) &:= \frac{\partial f_{dj}^{(\epsilon)}(t)}{\partial\epsilon}|_{\epsilon=0}=f_{dj}(0)\int_0^t\gamma_{dj}^{'}(u)dW(u),\\
A_{dj}^{(2)}(t) &:= \frac{1}{2}\frac{\partial^2 f_{dj}^{(\epsilon)}(t)}{\partial\epsilon^2}|_{\epsilon=0}=f_{dj}(0)\int_0^t\gamma_{dj}^{'}(u)\sum_{i=j+1}^{N}\left(\frac{-\tau_i f_{di}(0)}{1+\tau_i f_{di}(0)}\right)\gamma_{di}(u)du+\int_0^t A_{dj}^{(1)}(u)\gamma_{dj}^{'}(u)dW_u,
\end{aligned}$$

$$
\begin{aligned}
A_{fj}^{(1)}(t) &:= \frac{\partial f_{fj}^{(\epsilon)}(t)}{\partial \epsilon}|_{\epsilon=0} = f_{fj}(0)\int_0^t \gamma_{fj}^{'}(u)dW(u), \\
A_{fj}^{(2)}(t) &:= \frac{1}{2}\frac{\partial^2 f_{fj}^{(\epsilon)}(t)}{\partial \epsilon^2}|_{\epsilon=0} \\
&= f_{fj}(0)\int_0^t \gamma_{fj}^{'}(u)\left\{\sum_{i\in \hat{J}_{j+1}} -\left(\frac{-\tau_i f_{fi}(0)}{1+\tau_i f_{fi}(0)}\right)\gamma_{fi}(u) + \sum_{i\in \hat{J}_{N+1}}\left(\frac{-\tau_i f_{di}(0)}{1+\tau_i f_{di}(0)}\right)\gamma_{di}(u)\right\}du \\
&\quad - f_{fj}(0)\int_0^t \gamma_{fj}^{'}(u)\sigma(u)\bar{\sigma}du + \int_0^t A_{fj}^{(1)}(u)\gamma_{fj}^{'}(u)dW_u, \\
A_{\sigma}^{(1)}(t) &:= \frac{\partial \sigma^{(\epsilon)}(t)}{\partial \epsilon}|_{\epsilon=0} = Y_t\int_0^t Y_u^{-1}[\partial_\epsilon \mu(u)du + \omega^{'}(u)dW_u], \\
A_{\sigma}^{(2)}(t) &:= \frac{1}{2}\frac{\partial^2 \sigma^{(\epsilon)}(t)}{\partial \epsilon^2}|_{\epsilon=0} = \frac{1}{2}Y_t\int_0^t Y_u^{-1}[\partial_\epsilon^2\mu(u) + \partial_\sigma^2\mu(u)(A_\sigma^{(1)}(u))^2 + 2\partial_\epsilon\partial_\sigma\mu(u)A_\sigma^{(1)}(u)]du \\
&\quad + Y_t\int_0^t Y_u^{-1}A_\sigma^{(1)}(u)\partial_\sigma\omega^{'}(u)dW_u \\
&\textit{and} \\
Y_t &:= e^{\int_0^t \partial_\sigma \mu(u)du}.
\end{aligned}
$$

Here, the following notations are used;

$$
\begin{cases}
\sigma(u) \equiv \sigma^{(0)}(u), \quad \partial_\epsilon\mu(u) \equiv \partial_\epsilon\mu^{(\epsilon)}(u)|_{\epsilon=0}, \\
\partial_\epsilon^2\mu(u) \equiv \partial_\epsilon^2\mu^{(\epsilon)}(u)|_{\epsilon=0}, \quad \partial_\sigma\mu(u) \equiv \partial_\sigma\mu^{(\epsilon)}(u)|_{\epsilon=0}, \\
\partial_\sigma^2\mu(u) \equiv \partial_\sigma^2\mu^{(\epsilon)}(u)|_{\epsilon=0}, \quad \partial_\epsilon\partial_\sigma\mu(u) \equiv \partial_\epsilon\partial_\sigma\mu^{(\epsilon)}(u)|_{\epsilon=0}, \\
\omega(u) \equiv \omega(\sigma^{(0)}(u),u), \quad \partial_\sigma\omega(u) \equiv \partial_\sigma\omega(\sigma^{(\epsilon)}(u),u)|_{\epsilon=0}.
\end{cases} \tag{18}
$$

(*Proof*)
Only (15) is shown. (16) and (17) are obtained similarly. Differentiating the equation (9) with respect to ϵ once and twice, we have:

$$
\begin{aligned}
\frac{\partial f_{dj}^{(\epsilon)}(t)}{\partial \epsilon} &= 2\epsilon\int_0^t \left\{-f_{dj}^{(\epsilon)}(u)\gamma_{dj}^{'}(u)\sum_{i=j+1}^N \frac{\tau_i f_{di}^{(\epsilon)}(u)\gamma_{di}(u)}{1+\tau_i f_{di}^{(\epsilon)}(u)}\right\}du \\
&+ \epsilon^2\int_0^t \frac{\partial}{\partial\epsilon}\left\{-f_{dj}^{(\epsilon)}(u)\gamma_{dj}^{'}(u)\sum_{i=j+1}^N \frac{\tau_i f_{di}^{(\epsilon)}(u)\gamma_{di}(u)}{1+\tau_i f_{di}^{(\epsilon)}(u)}\right\}du \\
&+ \int_0^t f_{dj}^{(\epsilon)}(u)\gamma_{dj}^{'}(u)dW_u + \epsilon\int_0^t \left\{\frac{\partial}{\partial\epsilon}f_{dj}^{(\epsilon)}(u)\right\}\gamma_{dj}^{'}(u)dW_u \\
&\text{and} \\
\frac{\partial^2 f_{dj}^{(\epsilon)}(t)}{\partial \epsilon^2} &= 2\int_0^t \left\{-f_{dj}^{(\epsilon)}(u)\gamma_{dj}^{'}(u)\sum_{i=j+1}^N \frac{\tau_i f_{di}^{(\epsilon)}(u)\gamma_{di}(u)}{1+\tau_i f_{di}^{(\epsilon)}(u)}\right\}du \\
&+ 4\epsilon\int_0^t \frac{\partial}{\partial\epsilon}\left\{-f_{dj}^{(\epsilon)}(u)\gamma_{dj}^{'}(u)\sum_{i=j+1}^N \frac{\tau_i f_{di}^{(\epsilon)}(u)\gamma_{di}(u)}{1+\tau_i f_{di}^{(\epsilon)}(u)}\right\}du \\
&+ \epsilon^2\int_0^t \frac{\partial^2}{\partial\epsilon^2}\left\{-f_{dj}^{(\epsilon)}(u)\gamma_{dj}^{'}(u)\sum_{i=j+1}^N \frac{\tau_i f_{di}^{(\epsilon)}(u)\gamma_{di}(u)}{1+\tau_i f_{di}^{(\epsilon)}(u)}\right\}du \\
&+ 2\int_0^t \left\{\frac{\partial}{\partial\epsilon}f_{dj}^{(\epsilon)}(u)\right\}\gamma_{dj}^{'}(u)dW_u + \epsilon\int_0^t \left\{\frac{\partial^2}{\partial\epsilon^2}f_{dj}^{(\epsilon)}(u)\right\}\gamma_{dj}^{'}(u)dW_u.
\end{aligned}
$$

Then, setting $\epsilon = 0$, we obtain $A_{dj}^{(1)}(t)$ and $A_{dj}^{(2)}(t)$. □

Next, the following variables are defined:

$$
\begin{cases}
\sigma_X(u) := F_{N+1}(0)[\sum_{i\in \hat{J}_{N+1}} g_{fi}^{(0)}(u) - \sum_{i\in \hat{J}_{N+1}} g_{di}^{(0)}(u) + \sigma(u)\bar{\sigma}\ ] \\
g_{fi}^{(0)}(u) := \left(\frac{-\tau_i f_{fi}(0)}{1+\tau_i f_{fi}(0)}\right)\gamma_{fi}(u) \\
g_{di}^{(0)}(u) := \left(\frac{-\tau_i f_{di}(0)}{1+\tau_i f_{di}(0)}\right)\gamma_{di}(u) \\
g_{fi}^{(1)}(u) := \left(\frac{-\tau_i}{(1+\tau_i f_{fi}(0))^2}\right)\gamma_{fi}(u) \\
g_{di}^{(1)}(u) := \left(\frac{-\tau_i}{(1+\tau_i f_{di}(0))^2}\right)\gamma_{di}(u) \\
g_{fi}^{(2)}(u) := \left(\frac{2\tau_i^2}{(1+\tau_i f_{fi}(0))^3}\right)\gamma_{fi}(u) \\
g_{di}^{(2)}(u) := \left(\frac{2\tau_i^2}{(1+\tau_i f_{di}(0))^3}\right)\gamma_{di}(u)
\end{cases}
\tag{19}
$$

Then, the asymptotic expansion of a foreign exchange forward rate up to the third order of ϵ(ϵ^3-order) can be derived.

Proposition 1 *The asymptotic expansion of $F_{N+1}^{(\epsilon)}(t)$ up to the third order is expressed as follows:*

$$
F_{N+1}^{(\epsilon)}(t) = F_{N+1}(0) + \epsilon A_t^{(1)} + \epsilon^2 A_t^{(2)} + \epsilon^3 A_t^{(3)} + o(\epsilon^3) \tag{20}
$$

where

$$
\begin{aligned}
A_t^{(1)} &:= \int_0^t \sigma_X(u)' dW_u, && (21)\\
A_t^{(2)} &:= F_{N+1}(0)\int_0^t [\sum_{i\in \hat{J}_{N+1}} g_{fi}^{(1)}(u) A_{fi}^{(1)}(u) && (22)\\
&- \sum_{i\in \hat{J}_{N+1}} g_{di}^{(1)}(u) A_{di}^{(1)}(u) + A_\sigma^{(1)}(u)\bar{\sigma}]' dW_u \\
&+ \frac{1}{F_{N+1}(0)}\int_0^t A_u^{(1)} \sigma_X(u)' dW_u,
\end{aligned}
$$

and

$$
\begin{aligned}
A_t^{(3)} \quad := \quad & F_{N+1}(0) \sum_{i \in \hat{J}_{N+1}} \int_0^t A_{fi}^{(2)}(u) (g_{fi}^{(1)}(u))' dW_u \\
+ \quad & \frac{F_{N+1}(0)}{2} \sum_{i \in \hat{J}_{N+1}} \int_0^t (A_{fi}^{(1)}(u))^2 (g_{fi}^{(2)}(u))' dW_u \\
- \quad & F_{N+1}(0) \sum_{i \in \hat{J}_{N+1}} \int_0^t A_{di}^{(2)}(u) (g_{di}^{(1)}(u))' dW_u \\
- \quad & \frac{F_{N+1}(0)}{2} \sum_{i \in \hat{J}_{N+1}} \int_0^t (A_{di}^{(1)}(u))^2 (g_{di}^{(2)}(u))' dW_u \\
+ \quad & F_{N+1}(0) \int_0^t A_\sigma^{(2)}(u) \bar{\sigma}' dW_u \\
+ \quad & \sum_{i \in \hat{J}_{N+1}} \int_0^t (g_{fi}^{(1)}(u))' A_{fi}^{(1)}(u) A_u^{(1)} dW_u \\
- \quad & \sum_{i \in \hat{J}_{N+1}} \int_0^t (g_{di}^{(1)}(u))' A_{di}^{(1)}(u) A_u^{(1)} dW_u \\
+ \quad & \int_0^t A_\sigma^{(1)}(u) A_u^{(1)} \bar{\sigma}' dW_u \\
+ \quad & \frac{1}{F_{N+1}(0)} \int_0^t \sigma_X'(u) A_u^{(2)} dW_u.
\end{aligned}
\tag{23}
$$

(*Proof*)
We first note that

$$
F_{N+1}^{(\epsilon)}(t) \quad = \quad F_{N+1}(0) + \epsilon \frac{\partial F_{N+1}^{(\epsilon)}(t)}{\partial \epsilon}|_{\epsilon=0} + \frac{\epsilon^2}{2} \frac{\partial^2 F_{N+1}^{(\epsilon)}(t)}{\partial \epsilon^2}|_{\epsilon=0} + \frac{\epsilon^3}{6} \frac{\partial^3 F_{N+1}^{(\epsilon)}(t)}{\partial \epsilon^3}|_{\epsilon=0} + o(\epsilon^3),
$$

and set $A_t^{(1)} := \frac{\partial F_{N+1}^{(\epsilon)}(t)}{\partial \epsilon}|_{\epsilon=0}$, $A_t^{(2)} := \frac{1}{2} \frac{\partial^2 F_{N+1}^{(\epsilon)}(t)}{\partial \epsilon^2}|_{\epsilon=0}$ and $A_t^{(3)} := \frac{1}{6} \frac{\partial^3 F_{N+1}^{(\epsilon)}(t)}{\partial \epsilon^3}|_{\epsilon=0}$. As for (21), differentiating the equation (14) with respect to ϵ once, we have:

$$
\begin{aligned}
\frac{\partial F_{N+1}^{(\epsilon)}(t)}{\partial \epsilon} \quad = \quad & \int_0^t \left[\sum_{j \in \hat{J}_{N+1}} \frac{-\tau_j f_{fj}^{(\epsilon)}(u) \gamma_{fj}(u)}{1 + \tau_j f_{fj}^{(\epsilon)}(u)} - \sum_{j \in \hat{J}_{N+1}} \frac{-\tau_j f_{dj}^{(\epsilon)}(u) \gamma_{dj}(u)}{1 + \tau_i f_{dj}^{(\epsilon)}(u)} + \sigma^{(\epsilon)}(u) \bar{\sigma} \right]' F_{N+1}^{(\epsilon)}(u) dW_u \\
+ \quad & \epsilon \int_0^t \frac{\partial}{\partial \epsilon} \left[\sum_{j \in \hat{J}_{N+1}} \frac{-\tau_j f_{fj}^{(\epsilon)}(u) \gamma_{fj}(u)}{1 + \tau_j f_{fj}^{(\epsilon)}(u)} - \sum_{j \in \hat{J}_{N+1}} \frac{-\tau_j f_{dj}^{(\epsilon)}(u) \gamma_{dj}(u)}{1 + \tau_i f_{dj}^{(\epsilon)}(u)} + \sigma^{(\epsilon)}(u) \bar{\sigma} \right]' F_{N+1}^{(\epsilon)}(u) dW_u \\
+ \quad & \epsilon \int_0^t \left[\sum_{j \in \hat{J}_{N+1}} \frac{-\tau_j f_{fj}^{(\epsilon)}(u) \gamma_{fj}(u)}{1 + \tau_j f_{fj}^{(\epsilon)}(u)} - \sum_{j \in \hat{J}_{N+1}} \frac{-\tau_j f_{dj}^{(\epsilon)}(u) \gamma_{dj}(u)}{1 + \tau_i f_{dj}^{(\epsilon)}(u)} + \sigma^{(\epsilon)}(u) \bar{\sigma} \right]' \frac{\partial}{\partial \epsilon} \left\{ F_{N+1}^{(\epsilon)}(u) \right\} dW_u.
\end{aligned}
$$

Then, setting $\epsilon = 0$, and noting the definitions of $g_{fi}^{(0)}(u)$, $g_{di}^{(0)}(u)$, $\sigma(u)$ and $\sigma_X(u)$ in (18) and (19), we obtain the expression of $A_t^{(1)}$, that is (21).
Although tedious calculations are required, (22) and (23) can be obtained in the similar manner; we first differentiate the equation (14) with respect to ϵ twice and three times. Then, setting $\epsilon = 0$, substituting the expressions of $A_{dj}^{(1)}(t)$, $A_{dj}^{(2)}(t)$, $A_{fj}^{(1)}(t)$, $A_{fj}^{(2)}(t)$, $A_\sigma^{(1)}(t)$ and $A_\sigma^{(2)}(t)$ given in Lemma 1, and noting the definitions of $g_{fi}^{(1)}(u)$, $g_{di}^{(1)}(u)$, $g_{fi}^{(2)}(u)$, $g_{di}^{(2)}(u)$ and $\sigma_X(u)$ in (19), we obtain the expressions of $A_t^{(2)}$ and $A_t^{(3)}$.
□

With the expression of $F_{N+1}^{(\epsilon)}(t)$ in Proposition 1, we now focus on pricing options. Hereafter, we will consider a call option with strike rate K_ϵ where K_ϵ is defined for some arbitrary $y \in \mathbf{R}$ as

$$
K_\epsilon := F_{N+1}(0) - \epsilon y.
$$

Then, the discounted value of the option is given by

$$\begin{aligned}\frac{V(0;T_{N+1},K_\epsilon)}{P_d(0,T_{N+1})} &= \mathbf{E}[(F^{(\epsilon)}_{N+1}(T_{N+1})-K_\epsilon)^+] \\ &= \mathbf{E}[\epsilon(X^{(\epsilon)}+y)^+] \end{aligned} \tag{24}$$

where

$$X^{(\epsilon)} := \frac{F^{(\epsilon)}_{N+1}(T_{N+1})-F_{N+1}(0)}{\epsilon}. \tag{25}$$

Note that $X^{(\epsilon)}$ are expanded up to the third order as follows:

$$X^{(\epsilon)} = g_1+\epsilon g_2+\epsilon^2 g_3+o(\epsilon^2), \tag{26}$$

where

$$g_1 := A^{(1)}_{T_{N+1}} = \int_0^{T_{N+1}} \sigma_X(u)' dW_u, \tag{27}$$

$$\begin{aligned} g_2 &:= A^{(2)}_{T_{N+1}} = F_{N+1}(0)\int_0^{T_{N+1}}[\sum_{i\in\hat{J}_{N+1}} g^{(1)}_{fi}(u)A^{(1)}_{fi}(u) \qquad (28)\\ &- \sum_{i\in\hat{J}_{N+1}} g^{(1)}_{di}(u)A^{(1)}_{di}(u)+A^{(1)}_\sigma(u)\bar{\sigma}]' dW_u \\ &+ \frac{1}{F_{N+1}(0)}\int_0^{T_{N+1}} A^{(1)}_u \sigma_X(u)' dW_u, \end{aligned} \tag{29}$$

and

$$\begin{aligned} g_3 &:= A^{(3)}_{T_{N+1}} = F_{N+1}(0)\sum_{i\in\hat{J}_{N+1}}\int_0^{T_{N+1}} A^{(2)}_{fi}(u)(g^{(1)}_{fi}(u))' dW_u \\ &+ \frac{F_{N+1}(0)}{2}\sum_{i\in\hat{J}_{N+1}}\int_0^{T_{N+1}}(A^{(1)}_{fi}(u))^2(g^{(2)}_{fi}(u))' dW_u \\ &- F_{N+1}(0)\sum_{i\in\hat{J}_{N+1}}\int_0^{T_{N+1}} A^{(2)}_{di}(u)(g^{(1)}_{di}(u))' dW_u \\ &- \frac{F_{N+1}(0)}{2}\sum_{i\in\hat{J}_{N+1}}\int_0^{T_{N+1}}(A^{(1)}_{di}(u))^2(g^{(2)}_{di}(u))' dW_u \\ &+ F_{N+1}(0)\int_0^{T_{N+1}} A^{(2)}_\sigma(u)\bar{\sigma}' dW_u \\ &+ \sum_{i\in\hat{J}_{N+1}}\int_0^{T_{N+1}}(g^{(1)}_{fi}(u))' A^{(1)}_{fi}(u)A^{(1)}_u dW_u \\ &- \sum_{i\in\hat{J}_{N+1}}\int_0^{T_{N+1}}(g^{(1)}_{di}(u))' A^{(1)}_{di}(u)A^{(1)}_u dW_u \\ &+ \int_0^{T_{N+1}} A^{(1)}_\sigma(u)A^{(1)}_u\bar{\sigma}' dW_u \\ &+ \frac{1}{F_{N+1}(0)}\int_0^{T_{N+1}}\sigma'_X(u)A^{(2)}_u dW_u. \end{aligned} \tag{30}$$

Note also that the first order term g_1 follows normal distribution with mean 0 and variance Σ:

$$\Sigma := \int_0^{T_{N+1}}\sigma'_X(u)\sigma_X(u)du. \tag{31}$$

With the following theorem, an approximation of the density function of $F_{N+1}^{(\epsilon)}(T_{N+1})$ will be obtained.

Theorem 1 *Let $\phi_X^{(\epsilon)}(x)$ denote the probability density function of $X^{(\epsilon)}$. Then, under the assumption of $\Sigma > 0$, an asymptotic expansion of $\phi_X^{(\epsilon)}(x)$ is given by*

$$\begin{aligned}\phi_X^{(\epsilon)}(x) = \Bigg[1 \quad &+ \quad D_1^{(\epsilon)}\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right) + D_2^{(\epsilon)}\left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}\right) \\ &+ \quad D_3^{(\epsilon)}\left(\frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} + \frac{3}{\Sigma^2}\right) + D_4^{(\epsilon)}\left(\frac{x^5}{\Sigma^5} - \frac{10x^3}{\Sigma^4} + \frac{15x}{\Sigma^3}\right) \\ &+ \quad D_5^{(\epsilon)}\left(\frac{x^6}{\Sigma^6} - \frac{15x^4}{\Sigma^5} + \frac{45x^2}{\Sigma^4} - \frac{15}{\Sigma^3}\right)\Bigg] \times \phi_{0,\Sigma}(x) \\ + \quad o(\epsilon^2)\end{aligned} \tag{32}$$

where

$$\phi_{\mu,\Sigma}(x) := \frac{1}{\sqrt{2\pi\Sigma}} e^{\frac{-(x-\mu)^2}{2\Sigma}}$$

and

$$\begin{aligned}D_1^{(\epsilon)} &:= \epsilon C_{2,1} + \epsilon^2 C_{3,1} + \frac{1}{2}\epsilon^2 C_{4,0} \\ D_2^{(\epsilon)} &:= \epsilon C_{2,2} + \epsilon^2 C_{3,2} + \frac{1}{2}\epsilon^2 C_{4,1} \\ D_3^{(\epsilon)} &:= \epsilon^2 C_{3,3} + \frac{1}{2}\epsilon^2 C_{4,2} \\ D_4^{(\epsilon)} &:= \frac{1}{2}\epsilon^2 C_{4,3} \\ D_5^{(\epsilon)} &:= \frac{1}{2}\epsilon^2 C_{4,4}.\end{aligned}$$

All of $C_{2,1}, C_{2,2}, C_{3,1}, C_{3,2}, C_{3,3}, C_{4,0}, C_{4,1}, C_{4,2}, C_{4,3}$, and $C_{4,4}$ are constants and are defined in Appendix A.
(*Proof*)
Substituting $d = 1$, $\phi^{(\epsilon)}(x) \equiv 1$, and $B = (-\infty, x]$ in Theorem 3.4 of Kunitomo and Takahashi[2003], we can obtain an asymptotic expansion of the probability distribution function of $X^{(\epsilon)}$:

$$\begin{aligned}P\left(\{X^{(\epsilon)} \le x\}\right) \quad = \quad & \int_{-\infty}^{x} \phi_{0,\Sigma}(z)dz \\ & +\epsilon \int_{-\infty}^{x} -\frac{\partial}{\partial z}\left\{\mathbf{E}[g_2|g_1 = z]\phi_{0,\Sigma}(z)\right\}dz \\ & +\epsilon^2 \int_{-\infty}^{x} -\frac{\partial}{\partial z}\left\{\mathbf{E}[g_3|g_1 = z]\phi_{0,\Sigma}(z)\right\}dz \\ & +\frac{1}{2}\epsilon^2 \int_{-\infty}^{x} \frac{\partial^2}{\partial z^2}\left\{\mathbf{E}[g_2^2|g_1 = z]\phi_{0,\Sigma}(z)\right\}dz + o(\epsilon^2).\end{aligned}$$

Then, by differentiating both sides of the equation above with respect to x, we have:

$$\begin{aligned}\phi_X^{(\epsilon)}(x) \quad = \quad & \phi_{0,\Sigma}(x) \\ & -\epsilon\frac{\partial}{\partial x}\left\{\mathbf{E}[g_2|g_1 = x]\phi_{0,\Sigma}(x)\right\} \\ & -\epsilon^2\frac{\partial}{\partial x}\left\{\mathbf{E}[g_3|g_1 = x]\phi_{0,\Sigma}(x)\right\} \\ & +\frac{1}{2}\epsilon^2\frac{\partial^2}{\partial x^2}\left\{\mathbf{E}[g_2^2|g_1 = x]\phi_{0,\Sigma}(x)\right\} + o(\epsilon^2).\end{aligned}$$

Finally, noting that $\mathbf{E}[g_2|g_1 = x], \mathbf{E}[g_3|g_1 = x]$, and $\mathbf{E}[g_2^2|g_1 = x]$ are the following polynomials of x(see Appendix A for details.);

$$\mathbf{E}[g_2|g_1 = x] = C_{2,1}\frac{x}{\Sigma} + C_{2,2}(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}) \tag{33}$$

$$\mathbf{E}[g_3|g_1 = x] = C_{3,1}\frac{x}{\Sigma} + C_{3,2}(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}) + C_{3,3}(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}) \tag{34}$$

$$\begin{aligned}\mathbf{E}[g_2^2|g_1 = x] &= C_{4,0} + C_{4,1}\frac{x}{\Sigma} + C_{4,2}(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}) \\ &\quad + C_{4,3}(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}) + C_{4,4}(\frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} + \frac{3}{\Sigma^2}),\end{aligned} \tag{35}$$

we obtain the result. □

Remark 1 *In the following section, we set $\epsilon = 1$ for numerical examples. In that case, we easily notice $F^{(1)}_{N+1}(T_{N+1}) = X^{(1)} + F_{N+1}(0)$. Thus, the probability density function of $F^{(1)}_{N+1}(T_{N+1})$ is approximated as follows;*

$$\begin{aligned}\phi_F^{(1)}(x) := \Bigg[1 &+ D_1^{(1)}\left\{\frac{(x - F_{N+1}(0))^2}{\Sigma^2} - \frac{1}{\Sigma}\right\} \\ &+ D_2^{(1)}\left\{\frac{(x - F_{N+1}(0))^3}{\Sigma^3} - \frac{3(x - F_{N+1}(0))}{\Sigma^2}\right\} \\ &+ D_3^{(1)}\left\{\frac{(x - F_{N+1}(0))^4}{\Sigma^4} - \frac{6(x - F_{N+1}(0))^2}{\Sigma^3} + \frac{3}{\Sigma^2}\right\} \\ &+ D_4^{(1)}\left\{\frac{(x - F_{N+1}(0))^5}{\Sigma^5} - \frac{10(x - F_{N+1}(0))^3}{\Sigma^4} + \frac{15(x - F_{N+1}(0))}{\Sigma^3}\right\} \\ &+ D_5^{(1)}\left\{\frac{(x - F_{N+1}(0))^6}{\Sigma^6} - \frac{15(x - F_{N+1}(0))^4}{\Sigma^5} + \frac{45(x - F_{N+1}(0))^2}{\Sigma^4} - \frac{15}{\Sigma^3}\right\}\Bigg] \\ \times \quad \phi_{F_{N+1}(0),\Sigma}(x) \quad .\end{aligned} \tag{36}$$

Finally, an approximation formula for valuation of the European call option written on $F^{(\epsilon)}_{N+1}(T_{N+1})$ are stated.

Theorem 2 *We define $K_\epsilon := F_{N+1}(0) - \epsilon y$ for some arbitrary $y \in \mathbf{R}$ and suppose that $\Sigma > 0$. Then, an asymptotic expansion of $V(0; T_{N+1}, K_\epsilon)$, the value of the option with strike rate K_ϵ is given as follows:*

$$\begin{aligned}V(0; T_{N+1}, K_\epsilon) &= P_d(0, T_{N+1})\Bigg[\epsilon y\int_{-y}^{\infty}\phi_{0,\Sigma}(x)dx + \epsilon\int_{-y}^{\infty}x\phi_{0,\Sigma}(x)dx \\ &+ \epsilon^2\int_{-y}^{\infty}\mathbf{E}[g_2|g_1 = x]\phi_{0,\Sigma}(x)dx \\ &+ \epsilon^3\int_{-y}^{\infty}\mathbf{E}[g_3|g_1 = x]\phi_{0,\Sigma}(x)dx + \frac{\epsilon^3}{2}(\mathbf{E}[g_2^2|g_1 = x]\phi_{0,\Sigma}(x))_{x=-y}\Bigg] \\ &+ o(\epsilon^3),\end{aligned} \tag{37}$$

where $\mathbf{E}[g_2|g_1 = x]$, $\mathbf{E}[g_3|g_1 = x]$, and $\mathbf{E}[g_2^2|g_1 = x]$ are given in equations (33),(34), and (35).

(*Proof*)

With the equation in the proof of Theorem 1, we obtain:

$$
\begin{aligned}
\frac{V(0;T_{N+1},K_\epsilon)}{P_d(0,T_{N+1})} &= \mathbf{E}[\epsilon(X^{(\epsilon)}+y)^+] \\
&= \epsilon\int_{-y}^{\infty}(x+y)\phi_X^{(\epsilon)}(x)dx \\
&= \epsilon\int_{-y}^{\infty}(x+y)\phi_{0,\Sigma}(x)dx \\
&\quad -\epsilon^2\int_{-y}^{\infty}(x+y)\frac{\partial}{\partial x}\left\{\mathbf{E}[g_2|g_1=x]\phi_{0,\Sigma}(x)\right\}dx \\
&\quad -\epsilon^3\int_{-y}^{\infty}(x+y)\frac{\partial}{\partial x}\left\{\mathbf{E}[g_3|g_1=x]\phi_{0,\Sigma}(x)\right\}dx \\
&\quad +\frac{1}{2}\epsilon^3\int_{-y}^{\infty}(x+y)\frac{\partial^2}{\partial x^2}\left\{\mathbf{E}[g_2^2|g_1=x]\phi_{0,\Sigma}(x)\right\}dx+o(\epsilon^3).
\end{aligned}
$$

By applying integration-by-parts to the second, third and fourth terms of the right hand of the equation above and noting $E[g_2|g_1=x]$, $E[g_3|g_1=x]$ and $E[g_2^2|g_1=x]$ are the polynomials of x (see Appendix A for details.), we obtain the theorem. □

Remark 2 *Moreover, the following relations can be used to evaluate the integrals in the equation (37) of Theorem 2.*

$$
\begin{aligned}
\int_{-y}^{\infty}x\phi_{0,\Sigma}(x)dx &= \Sigma\phi_{0,\Sigma}(y) \\
\int_{-y}^{\infty}x^2\phi_{0,\Sigma}(x)dx &= \Sigma N\left(\frac{y}{\sqrt{\Sigma}}\right)-y\Sigma\phi_{0,\Sigma}(y) \\
\int_{-y}^{\infty}x^3\phi_{0,\Sigma}(x)dx &= (2\Sigma^2+\Sigma y^2)\phi_{0,\Sigma}(y) \\
\int_{-y}^{\infty}x^4\phi_{0,\Sigma}(x)dx &= 3\Sigma^2 N\left(\frac{y}{\sqrt{\Sigma}}\right)-(3\Sigma^2 y+\Sigma y^3)\phi_{0,\Sigma}(y)
\end{aligned}
$$

Remark 3 *In practice, we are often interested in the accuracy of our formulas for the prices of options whose underlying variables follow the SDEs (2), (5) and (6) with a particular set of parameters such as $\tilde{\gamma}_{dj}(t)$, $\tilde{\gamma}_{fj}(t)$, $\tilde{\sigma}(0)$, $\mu(t)$ and $\tilde{\omega}(\tilde{\sigma}(t),t)$. From this point of view, given some particular value of ϵ, $\gamma_{dj}(t)$, $\gamma_{fj}(t)$, $\sigma(0)$, $\mu^{(\epsilon)}(t)$ and $\omega(\sigma^{(\epsilon)}(t),t)$ in (9), (12) and (13) should be scaled so that $\epsilon\gamma_{dj}(t)=\tilde{\gamma}_{dj}(t)$, $\epsilon\gamma_{fj}(t)=\tilde{\gamma}_{fj}(t)$, $\epsilon\sigma(0)=\tilde{\sigma}(0)$, $\epsilon\mu^{(\epsilon)}(t)=\mu(t)$ and $\epsilon\omega(\sigma(t),t)=\tilde{\omega}(\tilde{\sigma}(t),t)$ for an arbitrary $t\in[0,T]$. For instance, $\gamma(t)$ is defined as $\gamma(t):=\frac{\tilde{\gamma}(t)}{\epsilon}$ where ϵ is fixed at a pre-specified constant through our procedure of expansions. Moreover, it can be shown that the approximated prices are unchanged whatever $\epsilon\in(0,1]$ is taken in evaluation, as long as above conditions are met. We here see this briefly.*

Suppose we rewrite the system of SDEs (9), (12), (13) and (14) not with ϵ but with another constant parameter $\delta\in(0,1]$. In order to guarantee that the above conditions are satisfied, δ must be written as $\delta=k\epsilon$ for some constant $k>0$ or equivalently, $\gamma_{dj}(t)$, $\gamma_{fj}(t)$, $\sigma(0)$, $\mu^{(\epsilon)}(t)$ and $\omega(\sigma^{(\epsilon)}(t),t)$ are replaced by $\frac{\gamma_{dj}(t)}{k}$, $\frac{\gamma_{fj}(t)}{k}$, $\frac{\sigma(0)}{k}$, $\frac{\mu^{(\epsilon)}(t)}{k}$, and $\frac{\omega(\sigma^{(\epsilon)}(t),t)}{k}$, respectively. Then, $C_{2,1}^{(k)}$, $C_{2,2}^{(k)}$, $C_{3,1}^{(k)}$, $C_{3,2}^{(k)}$, $C_{3,3}^{(k)}$, $C_{4,0}^{(k)}$, $C_{4,1}^{(k)}$, $C_{4,2}^{(k)}$, $C_{4,3}^{(k)}$, $C_{4,4}^{(k)}$ and $\Sigma^{(k)}$, the coefficients newly obtained and dependent on k, are given[1] *by*

$$
\begin{cases}
C_{2,1}^{(k)}=\frac{C_{2,1}}{k^3}, & C_{2,2}^{(k)}=\frac{C_{2,2}}{k^4}, \\
C_{3,1}^{(k)}=\frac{C_{3,1}}{k^4}, & C_{3,2}^{(k)}=\frac{C_{3,2}}{k^5}, \\
C_{3,3}^{(k)}=\frac{C_{3,3}}{k^6}, & \\
C_{4,0}^{(k)}=\frac{C_{4,0}}{k^4}, & C_{4,1}^{(k)}=\frac{C_{4,1}}{k^5}, \\
C_{4,2}^{(k)}=\frac{C_{4,2}}{k^6}, & C_{4,3}^{(k)}=\frac{C_{4,3}}{k^7}, \\
C_{4,4}^{(k)}=\frac{C_{4,4}}{k^8}, & \\
\Sigma^{(k)}=\frac{\Sigma}{k^2}. &
\end{cases}
$$

[1]Replace $\gamma_{dj}(t)$, $\gamma_{fj}(t)$, $\sigma^{(\epsilon)}(t)$, $\mu^{(\epsilon)}(t)$ and $\omega(\sigma^{(\epsilon)}(t),t)$ in derivations of these coefficients in Appendix A by $\frac{\gamma_{dj}(t)}{k}$, $\frac{\gamma_{fj}(t)}{k}$, $\frac{\sigma^{(\epsilon)}(t)}{k}$, $\frac{\mu^{(\epsilon)}(t)}{k}$ and $\frac{\omega(\sigma^{(\epsilon)}(t),t)}{k}$, respectively.

Table 1: Initial domestic/foreign forward interest rates and their volatilities

	f_d	γ_d^*	f_f	γ_f^*
case (i)	0.05	0.2	0.05	0.2
case (ii)	0.02	0.5	0.05	0.2
case (iii)	0.05	0.2	0.02	0.5
case (iv)	0.02	0.5	0.02	0.5

Substituting these coefficients into (33),(34), and (35), and replacing ϵ in (37) by δ, we can easily confirm that the approximations of the prices of the call options with strike prices K_ϵ and K_δ ($K_\epsilon = K_\delta$) in Theorem 2 will be the same. For instance,

$$\begin{aligned}
\delta^2 \int_{-y^{(\delta)}}^{\infty} \mathbf{E}[g_2^{(\delta)} | g_1^{(\delta)} = \hat{x}] \phi_{0,\Sigma^{(k)}}(\hat{x}) d\hat{x} &= \delta^2 \int_{-y^{(\delta)}}^{\infty} \left(C_{2,1}^{(k)} \frac{\hat{x}}{\Sigma^{(k)}} + C_{2,2}^{(k)} (\frac{\hat{x}^2}{(\Sigma^{(k)})^2} - \frac{1}{\Sigma^{(k)}}) \right) \phi_{0,\Sigma^{(k)}}(\hat{x}) d\hat{x} \\
&= (k\epsilon)^2 \int_{-y}^{\infty} \left(\frac{C_{2,1}}{k^3} \frac{x}{k} \frac{k^2}{\Sigma} + \frac{C_{2,2}}{k^4} (\frac{x^2}{k^2} \frac{k^4}{\Sigma^2} - \frac{k^2}{\Sigma}) \right) \phi_{0,\Sigma}(x) dx \\
&= \epsilon^2 \int_{-y}^{\infty} \left(C_{2,1} \frac{x}{\Sigma} + C_{2,2} (\frac{x}{\Sigma} - \frac{1}{\Sigma}) \right) \phi_{0,\Sigma}(x) dx \\
&= \epsilon^2 \int_{-y}^{\infty} \mathbf{E}[g_2 | g_1 = x] \phi_{0,\Sigma}(x) dx,
\end{aligned}$$

where $y^{(\delta)} = y/k$ and $g_i^{(\delta)}$ is an analogous variables g_i, $i = 1, 2$ in this section. This relationship for other terms in (37) can be shown similarly.

4 Numerical Examples

This section examines the effectiveness of our method through some numerical examples. First, the approximate probability density functions and option prices by our method are compared with their estimates by Monte Carlo simulations. Second, our pricing formula are applied to calibration of volatility surfaces observed in the JPY/USD currency option market.

4.1 Probability Densities and Option Prices

First of all, the processes of domestic and foreign forward interest rates and a volatility of the spot exchange rate are specified. We suppose $D = 4$, that is the dimension of Brownian motion is set to be four; it represents the uncertainty of domestic and foreign interest rates, the spot exchange rate, and its volatility. In particular, correlations among all factors are allowed.

Next, the volatility process (11) of the spot exchange rate under the domestic risk neutral measure are specified as follows;

$$\sigma^{(\epsilon)}(t) = \sigma(0) + \kappa \int_0^t (\theta - \sigma^{(\epsilon)}(u)) du + \epsilon \omega' \int_0^t \sqrt{\sigma^{(\epsilon)}(u)} d\hat{W} u \tag{38}$$

where θ and κ represent the level and speed of its mean-reversion respectively, and ω denotes a volatility on the volatility. In this section the parameters are set as follows. $\epsilon = 1$, $\sigma(0) = \theta = 0.1$, and $\kappa = 0.1$; $\omega = \omega^* V_\sigma$ where $\omega^* = 0.1$ and V_σ denotes a four dimensional constant vector given below.

It is further supposed that initial term structures of domestic and foreign forward interest rates are flat, and their volatilities have flat structures and are constant over time: that is, for all j, $f_{dj}(0) = f_d$, $f_{fj}(0) = f_f$, $\gamma_{dj}(t) := \gamma_d^* V_d 1_{[0,T_j)}(t)$ and $\gamma_{fj}(t) := \gamma_f^* V_f 1_{[0,T_j)}(t)$. Here, γ_d^* and γ_f^* are constant scalars, and V_d and V_f denote four dimensional constant vectors. Four cases for f_d, γ_d^*, f_f and γ_f^* as in Table 1 are considered. Moreover, given a correlation matrix $\underline{C}$ among all four factors, the constant vectors V_d, V_f, V_S and V_σ can be determined to satisfy $||V_d|| = ||V_f|| = ||V_S|| = ||V_\sigma|| = 1$ and $V'V = \underline{C}$ where $V := (V_d, V_f, V_S, V_\sigma)$, and $V_S \equiv \bar{\sigma}$; $\bar{\sigma}$ was defined right after the equation (4).

Finally, we make an assumption that $\gamma_{dn(t)-1}(t)$ and $\gamma_{fn(t)-1}(t)$, volatilities of the domestic and foreign interest rates applied to the period from t to the next fixing date $T_{n(t)}$, are set to be zero for arbitrary

$t \in [t, T_{n(t)}]$.

Figures 1-15 present probability density functions of $F^{(\epsilon)}_{N+1}(T_{N+1})$, the terminal value of the underlying asset, estimated by Monte Carlo simulations and our approximation formulas of the first, second and third orders with different maturities of 1, 2, 3, 4 and 5 years and with different correlation parameters: each estimate based on the Monte Carlo simulation is obtained by 1,000,000 trials with *antithetic variables method*. The three sets of correlation parameters are considered: In the case "Corr.1", all the factors are independent; In the case "Corr.2", there exists the correlation between the spot exchange rate and its volatility while there are no correlations among the others; In the case "Corr.3", the correlation between interest rates and the spot exchange rate are allowed while there are no correlations among the others. We also set $f_f = 0.05$, $\gamma^*_f = 0.2$, $f_d = 0.02$, and $\gamma^*_d = 0.5$ for interest rates as in the case (ii) of Table 1. Although experiments for the cases (i), (iii) and (iv) of Table 1 have been made, they are omitted because of similar results.

We notice that the approximations for the shorter maturities and the higher order approximations are closer to the probability density functions estimated by Monte Carlo simulations. While even the third order approximation tends to deviate in the tails of the densities for the longer maturities except for "Corr.2", in the region of high densities we can still observe the better fitness of the third order approximation than those of the approximations up to the second order. This may suggest that the higher order approximations increase the accuracy of our method.

Next, we show numerical examples for call option prices calculated by Monte Carlo simulations and our approximation formulas of the second and third orders with maturities of 1,2,3,4 and 5 years and with different correlation parameters as in the density functions. Moreover, the results of four different scenarios (i)-(iv) in table 1 for term structures of interest rates are shown. We set $K \sim F_{N+1}(0) \pm \sqrt{\Sigma}$ where Σ was defined in (31), and then moneyness, $\frac{K}{F_{N+1}(0)}$ for OTM and ITM are approximately given by $1 \pm \frac{\sqrt{\Sigma}}{F_{N+1}(0)}$ respectively. Thus, in the longer maturities the prices of the deeper OTM/ITM options are given in the tables, and especially the deepest OTM and ITM options are considered for the case "Corr.3" because the underlying forex forward is more volatile in that case than in the other cases. The summaries of the results are given in tables 2-5 and figures 16-33.

Tables 2-5 show the largest differences between the second(tables 2-3) or third(tables 4-5) orders approximations in the cases "Corr.1-3" and the estimates by Monte Carlo: Both the largest differences(indicating "diff.") and the largest relative(indicating "relative diff.") differences are listed, where "difference" and "relative difference" are defined by (the approximate value)-(the estimate by Monte Carlo) and (difference)/(the estimate by Monte Carlo)$\times$ 100(%), respectively. As for the largest relative differences, their levels(that is, "differences") are also given in tables 3 and 5.

Clearly, it can be observed that the third order approximations improve the second ones substantially; while the largest differences and the largest relative differences are -0.044(1y,"Corr.2") $\sim$ -0.502(5y,"Corr.1") and -5.29%(1y,"Corr.3") $\sim$ -18.13%(5y,"Corr.1") respectively in the second order, in the third order they are 0.008(1y,"Corr.3") $\sim$ -0.177(5y,"Corr.2") and 0.22%(1y,"Corr.3") $\sim$ -5.26%(3y,"Corr.2") respectively, where we also note in the table 5 that the level of -5.26%(3y,"Corr.2") is just -0.060. Moreover, it can be seen that the second order approximations undervalue the option prices, which seems corrected by the third order terms. Of course, in the other cases the deviations from the estimates by Monte Carlo simulations are smaller, which are not reported in the tables 2-5. (The details of the results will be given upon request.)

Figures 16-33 show the absolute values of the differences and relative differences between the third order approximations and the estimates by Monte Carlo simulations for the cases (i)-(iv) of "Corr. 1-3" in each ITM, ATM and OTM option. We note that the differences increase with longer maturities except for OTM options in relative differences while we can not observe particular patterns for the differences when interest rates' volatilities become high(low) as in the case (iv) (case (i)).

Further, it may be desirable in practice that the errors of approximations are smaller than bid-ask spreads observed in a currency option market. Hence, we examine whether our third order approximations satisfy this condition in the JPY/USD option market assuming that the estimates by Monte Carlo simulations are true option prices. The bid-ask spreads for ATM options in Feb. 19, 2007, Nov. 19, Aug. 19, and Feb. 19, 2006 are provided by Mizuho-DL Financial Technology Co., Ltd. as 0.07-0.11 for one year, 0.09-0.18 for two year and 0.12-0.42 for five year maturities, respectively. On the other hand, the absolute values of the deviations of the third order approximations from estimates by Monte Carlo simulations for ATM options are 0.000-0.012 for one year, 0.000-0.015 for two year and 0.010-0.063 for five year maturities, respectively. Thus, we can see that our third order approximations provide satisfactory results.

In addition to the experiments reported above, the accuracy of our method was also examined in a variety of cases for correlation parameters. Although these results are omitted due to their similarities to the previous ones, they will be given upon request.

4.2 Calibration to the Market

In this subsection, we calibrate our model parameters with the third order asymptotic expansion formula to observed volatilities with maturities from one to five years in the JPY/USD currency option market. Market makers in OTC currency option markets usually provide quotes on Black-Scholes implied volatilities and the moneyness of an option which is expressed in terms of Black-Scholes delta, rather than its strike price. The data of volatility surfaces on Jun 26 and July 5, 2006 are used, which consist of 25 delta put, 10 delta put, at-the-money, 10 delta call, and 25 delta call with their maturities of 1, 2, 3 and 5 years. We also construct term structures of domestic/foreign forward interest rates and volatilities using the data of swap rates and cap volatilities in each market (All data used in this subsection are provided by Mizuho-DL Financial Technology Co., Ltd.) .

Tables 6-7 and Figures 34-37 show the data on volatility surfaces and our calibrated parameters under the assumption of $\sigma(0) = \theta$ in (38); we make this assumption in order to avoid any paticular bias for θ, the level of a mean-reversion of the stochastic volatility, since forecasting θ is a usually difficult task (See the section 13.7 of Rebonato[2004] for the detail). Note that so-called a volatility skew can be found in the market, which may suggest a negative correlation between the spot exchange rate and its volatility. The parameters obtained by our calibration reflect this feature and seem to draw sufficiently accurate volatility surfaces on each date. The largest difference in these two dates is 0.19% and most of absolute values of differences are less than 0.15% where "difference" is defined by (the implied volatility calibrated by our formula)-(the implied volatility observed in the market).

Consequently, we conclude that our formula is flexible enough for the calibration of observed surfaces, and that we may use the calibrated parameters for valuation of illiquid options and more complex currency derivatives. Because calibrations are very time consuming by numerical methods such as Monte Carlo simulation, our closed-form formula seems quite useful in practice.

5 Conclusion

In this paper, we proposed approximation formulas based on an asymptotic expansion to evaluate currency options with a libor market model of domestic and foreign interest rates and stochastic volatility processes of spot exchange rates. We also provided numerical examples to investigate the accuracy of the approximations for the probability densities of currency forwards and option prices with maturities from one year to five years; in general, satisfactory results were obtained for the third order approximation up to five year maturity. Moreover, we applied the formula to the calibration of the JPY/USD currency option market and had successful results in reconstructing observed volatility surfaces.

Finally, our research plans are stated as follows: Similar methods will be applied to valuation and calibration of options with longer maturities; higher order asymptotic expansions or/and different types of expansions may be used. Asymptotic expansion formulas will be utilized for extended models where stochastic volatility structures of interest rates are allowed or/and a jump component is added to the volatility process of the spot exchange rate. In fact, some results will appear in a subsequent paper(Takahashi and Takehara[2008]).

6 Appendix

A Coefficients in the Asymptotic Expansion

This section presents the expressions of coefficients $C_{2,1}$, $C_{2,2}$, $C_{3,1}$, $C_{3,2}$, $C_{3,3}$, $C_{4,0}$, $C_{4,1}$, $C_{4,2}$, $C_{4,3}$ and $C_{4,4}$ in Theorem 1. First, they are shown as relatively compact forms:

$$
\left\{
\begin{aligned}
&C_{2,1} := c_2 \\
&C_{2,2} := \left[\sum_{i\in\hat{J}_{N+1}} \left(a^f_{2i} - a^d_{2i}\right)\right] + b_2 + d_2 \\
&C_{3,1} := \left[\sum_{i\in\hat{J}_{N+1}} \left\{(a^f_{3i} + c^f_{3i} + i^f_{3i}) - (a^d_{3i} + c^d_{3i} + i^d_{3i})\right\}\right] + e_3 + k_3 \\
&C_{3,2} := f_3 + l_3 + o_3 \\
&C_{3,3} := \left[\sum_{i\in\hat{J}_{N+1}} \left\{(b^f_{3i} + d^f_{3i} + j^f_{3i} + n^f_{3i}) - (b^d_{3i} + d^d_{3i} + j^d_{3i} + n^d_{3i})\right\}\right] + h_3 + m_3 + p_3 + q_3 \\
&C_{4,0} := \left[\sum_{i,j\in\hat{J}_{N+1}} (c^f_{4i,j} + c^d_{4i,j})\right] + e_4 + k_4 + n_4 \\
&\qquad\qquad - \left[\sum_{i,j\in\hat{J}_{N+1}} q_{4i,j}\right] + \left[\sum_{i\in\hat{J}_{N+1}} \left\{(v^f_{4i} + y^f_{4i}) - (v^d_{4i} + y^d_{4i})\right\}\right] + ad_4 \\
&C_{4,1} := h_4 - \left[\sum_{i\in\hat{J}_{N+1}} \left(s^f_{4i} - s^d_{4i}\right)\right] + aa_4 \\
&C_{4,2} := \left[\sum_{i,j\in\hat{J}_{N+1}} \left(b^f_{4i,j} + b^d_{4i,j}\right)\right] + d_4 + j_4 + m_4 \\
&\qquad\qquad - \left[\sum_{i,j\in\hat{J}_{N+1}} p_{4i,j}\right] + \left[\sum_{i\in\hat{J}_{N+1}} \left\{(u^f_{4i} + x^f_{4i}) - (u^d_{4i} + x^d_{4i})\right\}\right] + ac_4 \\
&C_{4,3} := f_4 - \left[\sum_{i\in\hat{J}_{N+1}} \left(r^f_{4i} - r^d_{4i}\right)\right] + z_4 \\
&C_{4,4} := \left[\sum_{i,j\in\hat{J}_{N+1}} \left(a^f_{4i,j} + a^d_{4i,j}\right)\right] + i_4 + l_4 \\
&\qquad\qquad - \left[\sum_{i,j\in\hat{J}_{N+1}} o_{4i,j}\right] + \left[\sum_{i\in\hat{J}_{N+1}} \left\{(t^f_{4i} + w^f_{4i}) - (t^d_{4i} + w^d_{4i})\right\}\right] + ab_4
\end{aligned}
\right. \tag{39}
$$

Subsections A.1 and A.2 below provide the expressions for the terms on the right hand side of (39); the expressions for the terms with superscript "f " such as a^f_{2i}, a^f_{3i} or $a^f_{4i,j}$ explicitly are not shown because they are obtained in the same manner as those for the terms with superscript "d " .(The details will be given upon request.)

Next, Let q_1, q_2, q_3, q_4 and q_5 denote the $\mathbf{R}^d$-valued functions on t and q_1 be $q_{1t} := \sigma_X(t)$. Then, the following functionals are defined:

$$
\begin{aligned}
I_1^1(q_2;T) &:= \int_0^T q_{2t}' q_{1t} dt \\
I_2^2(q_2,q_3;T) &:= \int_0^T q_{3t}' q_{1t} \int_0^t q_{2u}' q_{1u} du dt \\
I_0^3(q_2,q_3;T) &:= \int_0^T q_{2t}' q_{3t} dt \\
I_2^3(q_2,q_3;T) &:= \left(\int_0^T q_{2t}' q_{1t} dt\right)\left(\int_0^T q_{3t}' q_{1t} dt\right) \\
I_3^4(q_2,q_3,q_4;T) &:= \int_0^T q_{4t}' q_{1t} \int_0^t q_{3u}' q_{1u} \int_0^u q_{2s}' q_{1s} ds du dt \\
I_1^5(q_2,q_3,q_4;T) &:= \int_0^T q_{4t}' q_{1t} \int_0^t q_{2u}' q_{3u} du dt \\
I_3^5(q_2,q_3,q_4;T) &:= \int_0^T q_{4t}' q_{1t} \left(\int_0^t q_{2u}' q_{1u} du\right)\left(\int_0^t q_{3u}' q_{1u} du\right) dt \\
I_1^6(q_2,q_3,q_4;T) &:= \int_0^T q_{3t}' q_{1t} \int_0^t q_{2u}' q_{4u} du dt + \int_0^T q_{3t}' q_{4t} \int_0^t q_{2u}' q_{1u} du dt \\
I_3^6(q_2,q_3,q_4;T) &:= \left(\int_0^T q_{3t}' q_{1t} \int_0^t q_{2u}' q_{1u} du dt\right)\left(\int_0^T q_{4t}' q_{1t} dt\right) \\
I_0^7(q_2,q_3,q_4,q_5;T) &:= \int_0^T q_{3t}' q_{5t} \int_0^t q_{2u}' q_{4u} du dt \\
I_2^7(q_2,q_3,q_4,q_5;T) &:= \int_0^T q_{3t}' q_{1t} \int_0^t q_{5u}' q_{1u} \int_0^u q_{2s}' q_{4s} ds du dt \\
&\quad + \int_0^T q_{5t}' q_{1t} \int_0^t q_{3u}' q_{1u} \int_0^u q_{2s}' q_{4s} ds du dt \\
&\quad + \int_0^T q_{3t}' q_{1t} \int_0^t q_{2u}' q_{5u} \int_0^u q_{4s}' q_{1s} ds du dt \\
&\quad + \int_0^T q_{3t}' q_{5t} \left(\int_0^t q_{2u}' q_{1u} du\right)\left(\int_0^t q_{4u}' q_{1u} du\right) dt \\
&\quad + \int_0^T q_{5t}' q_{1t} \int_0^t q_{3u}' q_{4u} \int_0^u q_{2s}' q_{1s} ds du dt \\
I_4^7(q_2,q_3,q_4,q_5;T) &:= \left(\int_0^T q_{3t}' q_{1t} \int_0^t q_{2u}' q_{1u} du dt\right)\left(\int_0^T q_{5t}' q_{1t} \int_0^t q_{4u}' q_{1u} du dt\right)
\end{aligned}
$$

I_k^l, with its superscript l and subscript k, corresponds to the coefficient of the kth order polynomial of x in *Formula* l given in Appendix B.

It is stressed that most coefficients are expressed as linear combinations of only a dozen of different functionals defined above, and that this seems to make it easy to implement our method.

A.1 The Second Order

In this subsection, we concentrate on the second order scheme. First, we note that g_1 and g_2 are expressed as

$$
\begin{aligned}
g_1 &= A^{(1)}_{T_{N+1}} = \int_0^{T_{N+1}} \sigma_X(u)' dW_u \\
g_2 &= A^{(2)}_{T_{N+1}} = F_{N+1}(0) \int_0^{T_{N+1}} [\sum_{i \in \hat{J}_{N+1}} g^{(1)}_{fi}(u) A^{(1)}_{fi}(u) \\
&- \sum_{i \in \hat{J}_{N+1}} g^{(1)}_{di}(u) A^{(1)}_{di}(u) + A^{(1)}_{\sigma}(u)\bar{\sigma}]' dW_u \\
&+ \frac{1}{F_{N+1}(0)} \int_0^{T_{N+1}} A^{(1)}_u \sigma_X(u)' dW_u.
\end{aligned}
$$

Let $T \equiv T_{N+1}$, $F(0) \equiv F_{N+1}(0)$ and define $I_\mu(t) := \int_0^t Y_u^{-1} \partial_\epsilon \mu(u) du$ to avoid complex expressions. Then,

$$
\begin{aligned}
\mathbf{E}[g_2 | g_1 = x] &= F(0)\mathbf{E}\left[\int_0^T \sum_{i \in \hat{J}_{N+1}} A^{(1)}_{fi}(u)(g^{(1)}_{fi}(u))' dW_u | g_1 = x \right] \\
&- F(0)\mathbf{E}\left[\int_0^T \sum_{i \in \hat{J}_{N+1}} A^{(1)}_{di}(u)(g^{(1)}_{di}(u))' dW_u | g_1 = x \right] \\
&+ F(0)\mathbf{E}\left[\int_0^T A^{(1)}_{\sigma}(u)\bar{\sigma}' dW_u | g_1 = x \right] \\
&+ \frac{1}{F(0)}\mathbf{E}\left[\int_0^T A^{(1)}_u \sigma_X(u)' dW_u | g_1 = x \right].
\end{aligned}
$$

To evaluate the right hand side of the equation above, we utilize some formulas associated with conditional expectations of Gaussianity: The formulas are listed in Appendix B. In particular, applying (1) and (2) in Appendix B, we can evaluate each term in $\mathbf{E}[g_2 | g_1 = x]$ with $I_1^1(\cdot; T)$ and $I_2^2(\cdot, \cdot; T)$ as follows:

1.
$$
\begin{aligned}
F(0)\mathbf{E}\left[\int_0^T A^{(1)}_{di}(u)(g^{(1)}_{di}(u))' dW_u | g_1 = x \right] &= F(0)\left(\frac{-\tau_i f_{di}(0)}{(1 + \tau_i f_{di}(0))^2} \right) I_2^2(\gamma_{di}, \gamma_{di}; T) \times \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) \\
&=: a^d_{2i} \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right)
\end{aligned}
$$

2.
$$
\begin{aligned}
\frac{1}{F(0)}\mathbf{E}\left[\int_0^T A^{(1)}_u \sigma_X(u)' dW_u | g_1 = x \right] &= \frac{1}{F(0)} I_2^2(\sigma_X, \sigma_X; T) \times \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) \\
&=: b_2 \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right)
\end{aligned}
$$

3.
$$
\begin{aligned}
& F(0)\mathbf{E}\left[\int_0^T A^{(1)}_{\sigma}(u)\bar{\sigma}' dW_u | g_1 = x \right] \\
=\ & F(0) \left[I_1^1 \left(I_\mu Y \bar{\sigma}; T \right) \times \frac{x}{\Sigma} + I_2^2 \left(Y^{-1}\omega, Y\bar{\sigma}; T \right) \times \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) \right] \\
=:\ & c_2 \frac{x}{\Sigma} + d_2 \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right)
\end{aligned}
$$

Then, $C_{2,1}$ and $C_{2,2}$ are defined by

$$
\begin{aligned}
C_{2,1} &= c_2 \\
C_{2,2} &= \left[\sum_{i \in \hat{J}_{N+1}} \left(a^f_{2i} - a^d_{2i} \right) \right] + b_2 + d_2.
\end{aligned}
$$

A.2 The Third Order

A.2.1 Computation of $\mathbf{E}[g_3|g_1 = x]$

We first note that

$$
\begin{aligned}
g_3 = A_T^{(3)} &= F(0)\int_0^T \sum_{i\in\hat{J}_{N+1}} A_{fi}^{(2)}(u)(g_{fi}^{(1)}(u))^{'}dW_u \\
&+ \frac{F(0)}{2}\int_0^T \sum_{i\in\hat{J}_{N+1}} (A_{fi}^{(1)}(u))^2(g_{fi}^{(2)}(u))^{'}dW_u \\
&- F(0)\int_0^T \sum_{i\in\hat{J}_{N+1}} A_{di}^{(2)}(u)(g_{di}^{(1)}(u))^{'}dW_u \\
&- \frac{F(0)}{2}\int_0^T \sum_{i\in\hat{J}_{N+1}} (A_{di}^{(1)}(u))^2(g_{di}^{(2)}(u))^{'}dW_u \\
&+ F(0)\int_0^T A_{\sigma}^{(2)}(u)\bar{\sigma}^{'}dW_u \\
&+ \int_0^T \sum_{i\in\hat{J}_{N+1}} (g_{fi}^{(1)}(u))^{'}A_{fi}^{(1)}(u)A_u^{(1)}dW_u \\
&- \int_0^T \sum_{i\in\hat{J}_{N+1}} (g_{di}^{(1)}(u))^{'}A_{di}^{(1)}(u)A_u^{(1)}dW_u \\
&+ \int_0^T A_{\sigma}^{(1)}(u)A_u^{(1)}\bar{\sigma}^{'}dW_u \\
&+ \frac{1}{F(0)}\int_0^T \sigma_X^{'}(u)A_u^{(2)}dW_u
\end{aligned}
$$

Define $C_{dj}^{(2)}(u)$ and $C_{fj}^{(2)}(u)$ as

$$
\begin{aligned}
C_{dj}^{(2)}(u) &:= f_{dj}(0)\int_0^u \gamma_{dj}^{'}(s)\sum_{i=j+1}^{N}\left(\frac{-\tau_i f_{di}(0)}{1+\tau_i f_{di}(0)}\right)\gamma_{di}(s)ds, \\
C_{fj}^{(2)}(u) &:= f_{fj}(0)\int_0^u \gamma_{fj}^{'}(s)\left\{\sum_{i\in\hat{J}_{j+1}} -\left(\frac{-\tau_i f_{fi}(0)}{1+\tau_i f_{fi}(0)}\right)\gamma_{fi}(s) + \sum_{i\in\hat{J}_{N+1}}\left(\frac{-\tau_i f_{di}(0)}{1+\tau_i f_{di}(0)}\right)\gamma_{di}(s)\right\}ds \\
&- f_{fj}(0)\int_0^u \gamma_{fj}^{'}(s)\sigma^{(0)}(s)\bar{\sigma}ds.
\end{aligned}
$$

Then, we take the expectation of each term of g_3 conditional to $g_1 = x$. To evaluate each expectation, we use formulas in Appendix B, again. Results are reported below;

1. Apply *formulas 1,4.*

$$
\begin{aligned}
& F(0)\mathbf{E}\left[\int_0^T A_{di}^{(2)}(u)(g_{di}^{(1)}(u))^{'}dW_u|g_1 = x\right] \\
=\ & F(0)\left(\frac{-\tau_i}{(1+\tau_i f_{di}(0))^2}\right)\left[I_1^1(C_{di}^{(2)}\times\gamma_{di};T)\times\frac{x}{\Sigma} + f_{di}(0)I_3^4(\gamma_{di},\gamma_{di},\gamma_{di};T)\times\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)\right] \\
=:\ & a_{3i}^d\frac{x}{\Sigma} + b_{3i}^d\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)
\end{aligned}
$$

2. Apply *formula 5.*

$$
\begin{aligned}
& \frac{F(0)}{2}\mathbf{E}\left[\int_0^T (A_{di}^{(1)}(u))^2(g_{di}^{(2)}(u))^{'}dW_u|g_1 = x\right] \\
=\ & \frac{F(0)}{2}(f_{di}(0))^2\left(\frac{2\tau_i^2}{(1+\tau_i f_{di}(0))^3}\right)\left[I_1^5(\gamma_{di},\gamma_{di},\gamma_{di};T)\times\frac{x}{\Sigma} + I_3^5(\gamma_{di},\gamma_{di},\gamma_{di};T)\times\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)\right] \\
=:\ & c_{3i}^d\frac{x}{\Sigma} + d_{3i}^d\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)
\end{aligned}
$$

3. Apply *formulas 1,2,4,5.*

$$
\begin{aligned}
& F(0)\mathbf{E}\left[\int_0^T A_\sigma^{(2)}(u)\bar{\sigma}dW_u|g_1=x\right] \\
= \quad & F(0)\left[\left\{\frac{1}{2}\int_0^T\left(\int_0^u Y_s^{-1}\partial_\epsilon^2\mu_s ds\right)Y_u\bar{\sigma}'\sigma_X(u)du\right\}\times\frac{x}{\Sigma}\right] \\
+ \quad & F(0)\left[\left\{\frac{1}{2}\int_0^T\left(\int_0^u Y_s\partial_\sigma^2\mu_s\left(\int_0^s Y_\tau^{-1}\partial_\epsilon\mu_\tau d\tau\right)^2 ds\right)Y_u\bar{\sigma}'\sigma_X(u)du\right\}\times\frac{x}{\Sigma}\right. \\
& +\left\{\int_0^T\left(\int_0^u Y_s\partial_\sigma^2\mu_s\left(\int_0^s Y_\tau^{-1}\partial_\epsilon\mu_\tau d\tau\right)\left(\int_0^s Y_\tau^{-1}\omega'(\tau)\sigma_X(\tau)d\tau\right)ds\right)Y_u\bar{\sigma}'\sigma_X(u)du\right\} \\
& \times\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right) \\
& +\frac{1}{2}\left\{\int_0^T\left(\int_0^u Y_s\partial_\sigma^2\mu_s\left(\int_0^s Y_\tau^{-2}\omega'(\tau)\omega(\tau)d\tau\right)ds\right)Y_u\bar{\sigma}'\sigma_X(u)du\right\}\frac{x}{\Sigma} \\
& \left.+\frac{1}{2}\left\{\int_0^T\left(\int_0^u Y_s\partial_\sigma^2\mu_s\left(\int_0^s Y_\tau^{-1}\omega'(\tau)\sigma_X(\tau)d\tau\right)^2 ds\right)Y_u\bar{\sigma}'\sigma_X(u)du\right\}\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)\right] \\
+ \quad & F(0)\left[\left\{\int_0^T\left(\int_0^u \partial_\epsilon\partial_\sigma\mu_s\left(\int_0^s Y_\tau^{-1}\partial_\epsilon\mu_\tau d\tau\right)ds\right)Y_u\bar{\sigma}'\sigma_X(u)du\right\}\frac{x}{\Sigma}\right. \\
& \left.+\left\{\int_0^T\left(\int_0^u \partial_\epsilon\partial_\sigma\mu_s\left(\int_0^s Y_\tau^{-1}\omega'(\tau)\sigma_X(\tau)d\tau\right)ds\right)Y_u\bar{\sigma}'\sigma_X(u)du\right\}\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right)\right] \\
+ \quad & F(0)\left[\left\{\int_0^T\left(\int_0^u\left(\int_0^s Y_\tau^{-1}\partial_\epsilon\mu_\tau d\tau\right)\partial_\sigma\omega'(s)\sigma_X(s)ds\right)Y_u\bar{\sigma}'\sigma_X(u)du\right\}\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right)\right. \\
& \left.+\left\{\int_0^T\left(\int_0^u\left(\int_0^s Y_\tau^{-1}\omega'(\tau)\sigma_X(\tau)d\tau\right)\partial_\sigma\omega'(s)\sigma_X(s)ds\right)Y_u\bar{\sigma}'\sigma_X(u)du\right\}\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)\right] \\
=: \quad & e_3\frac{x}{\Sigma}+f_3\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right)+h_3\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)
\end{aligned}
$$

4. Apply *formula 5.*

$$
\begin{aligned}
& \mathbf{E}\left[\int_0^T (g_{di}^{(1)}(u))'A_{di}^{(1)}(u)A_u^{(1)}dW_u|g_1=x\right] \\
= \quad & \left(\frac{-\tau_i f_{di}(0)}{(1+\tau_i f_{di}(0))^2}\right)\times \\
& \left[I_1^5(\gamma_{di},\sigma_X,\gamma_{di};T)\times\frac{x}{\Sigma}+I_3^5(\gamma_{di},\sigma_X,\gamma_{di};T)\times\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)\right] \\
=: \quad & i_{3i}^d\frac{x}{\Sigma}+j_{3i}^d\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)
\end{aligned}
$$

5. Apply *formulas 2,5.*

$$
\begin{aligned}
& \mathbf{E}\left[\int_0^T A_\sigma^{(1)}(u)A_u^{(1)}\bar{\sigma}'dW_u|g_1=x\right] \\
= \quad & I_2^2(\sigma_X,I_\mu Y\bar{\sigma};T)\times\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right) \\
& +I_1^5(Y^{-1}\omega,\sigma_X,Y\bar{\sigma};T)\times\frac{x}{\Sigma}+I_3^5(Y^{-1}\omega,\sigma_X,Y\bar{\sigma};T)\times\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right) \\
=: \quad & k_3\frac{x}{\Sigma}+l_3\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right)+m_3\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)
\end{aligned}
$$

6. We first note that $\frac{1}{F(0)}\int_0^T \sigma_X^{'}(u)A_u^{(2)}dW_u$ is expressed as follows:

$$
\begin{aligned}
& \frac{1}{F(0)}\int_0^T \sigma_X^{'}(u)A_u^{(2)}dW_u \\
= & \sum_{j\in\hat{J}_{N+1}}\int_0^T\left(\int_0^u A_{fi}^{(1)}(s)(g_{fi}^{(1)}(s))^{'}dW_s\right)\sigma_X^{'}(u)dW_u \\
- & \sum_{j\in\hat{J}_{N+1}}\int_0^T\left(\int_0^u A_{di}^{(1)}(s)(g_{di}^{(1)}(s))^{'}dW_s\right)\sigma_X^{'}(u)dW_u \\
+ & \int_0^T\left(\int_0^u A_{\sigma}^{(1)}(s)\bar{\sigma}^{'}dW_u\right)\sigma_X^{'}(u)dW_u \\
+ & \frac{1}{F(0)^2}\int_0^T\left(\int_0^u A^{(1)}(s)\sigma_X^{'}(s)dW_s\right)\sigma_X^{'}(u)dW_u
\end{aligned}
$$

Then,

- Apply *formula 4*.

$$
\begin{aligned}
& \mathbf{E}\left[\int_0^T\left(\int_0^u A_{di}^{(1)}(s)(g_{di}^{(1)}(s))^{'}dW_s\right)\sigma_X^{'}(u)dW_u|g_1=x\right] \\
= & \left(\frac{-\tau_i f_{di}(0)}{(1+\tau_i f_{di}(0))^2}\right)I_3^4(\gamma_{di},\gamma_{di},\sigma_X;T)\times\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right) \\
=: & \; n_{3i}^d\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)
\end{aligned}
$$

- Apply *formulas 2,4*.

$$
\begin{aligned}
& \mathbf{E}\left[\int_0^T\left(\int_0^u A_{\sigma}^{(1)}(s)\bar{\sigma}^{'}dW_u\right)\sigma_X^{'}(u)dW_u|g_1=x\right] \\
= & \; I_2^2\left(I_\mu Y\bar{\sigma},\sigma_X;T\right)\times\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right) \\
& +I_3^4(Y^{-1}\omega,Y\bar{\sigma},\sigma_X;T)\times\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right) \\
=: & \; o_3\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}\right)+p_3\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)
\end{aligned}
$$

- Apply *formula 4*.

$$
\begin{aligned}
& \frac{1}{F(0)^2}\mathbf{E}\left[\int_0^T\left(\int_0^u A^{(1)}(s)\sigma_X^{'}(s)dW_s\right)\sigma_X^{'}(u)dW_u|g_1=x\right] \\
= & \frac{1}{F(0)^2}\times I_3^4(\sigma_X,\sigma_X,\sigma_X;T)\times\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right) \\
=: & \; q_3\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)
\end{aligned}
$$

Finally, coefficients of $C_{3,1}$, $C_{3,2}$, and $C_{3,3}$ can be defined as follows;

$$
\begin{aligned}
C_{3,1} &= \left[\sum_{i\in\hat{J}_{N+1}}\left\{(a_{3i}^f+c_{3i}^f+i_{3i}^f)-(a_{3i}^d+c_{3i}^d+i_{3i}^d)\right\}\right]+e_3+k_3 \\
C_{3,2} &= f_3+l_3+o_3 \\
C_{3,3} &= \left[\sum_{i\in\hat{J}_{N+1}}\left\{(b_{3i}^f+d_{3i}^f+j_{3i}^f+n_{3i}^f)-(b_{3i}^d+d_{3i}^d+j_{3i}^d+n_{3i}^d)\right\}\right]+h_3+m_3+p_3+q_3.
\end{aligned}
$$

A.2.2 Computation of $\mathbf{E}[g_2^2|g_1 = x]$

First, note that g_2^2 is expressed as

$$g_2^2 = \Bigg(F(0)\int_0^T [\sum_{i\in\hat{J}_{N+1}} g_{fi}^{(1)}(u)A_{fi}^{(1)}(u) - \sum_{i\in\hat{J}_{N+1}} g_{di}^{(1)}(u)A_{di}^{(1)}(u) + A_{\sigma}^{(1)}(u)\bar{\sigma}]' dW_u$$
$$+\frac{1}{F(0)}\int_0^T A_u^{(1)}\sigma_X(u)' dW_u \Bigg)^2 .$$

Next, we easily notice that $\mathbf{E}[g_2^2|g_1 = x]$ consists of the following terms.

1. Apply *formula 7.*

$$\begin{aligned}
& F(0)^2\mathbf{E}\left[\left(\int_0^T g_{di}^{(1)}(u)A_{di}^{(1)}(u)dW_u\right)\left(\int_0^T g_{dj}^{(1)}(u)A_{dj}^{(1)}(u)dW_u\right)|g_1 = x\right] \\
= \;& F(0)^2\left(\frac{-\tau_i f_{di}(0)}{(1+\tau_i f_{di}(0))^2}\right)\left(\frac{-\tau_j f_{dj}(0)}{(1+\tau_j f_{dj}(0))^2}\right)\times \\
& \Bigg[I_4^7(\gamma_{di},\gamma_{di},\gamma_{dj},\gamma_{dj};T)\times\left(\frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} + \frac{3}{\Sigma^2}\right) \\
& \quad +I_2^7(\gamma_{di},\gamma_{di},\gamma_{dj},\gamma_{dj};T)\times\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma^2}\right) + I_0^7(\gamma_{di},\gamma_{di},\gamma_{dj},\gamma_{dj};T)\Bigg] \\
=: \;& a_{4i,j}^d\left(\frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} + \frac{3}{\Sigma^2}\right) + b_{4i,j}^d\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma^2}\right) + c_{4i,j}^d
\end{aligned}$$

2.

$$\begin{aligned}
& F(0)^2\mathbf{E}\left[\left(\int_0^T A_{\sigma}^{(1)}(u)\bar{\sigma}' dW_u\right)^2 |g_1 = x\right] \\
= \;& F(0)^2\mathbf{E}\left[\left(\int_0^T I_{\mu}(u)Y_u\bar{\sigma}' dW_u\right)^2 |g_1 = x\right] \\
+ \;& 2F(0)^2\mathbf{E}\left[\left(\int_0^T I_{\mu}(u)Y_u\bar{\sigma}' dW_u\right)\left(\int_0^T\left(\int_0^u Y_s^{-1}\omega'(s)dW_s\right)Y_u\bar{\sigma}' dW_u\right)|g_1 = x\right] \\
+ \;& F(0)^2\mathbf{E}\left[\left(\int_0^T\left(\int_0^u Y_s^{-1}\omega'(s)dW_s\right)Y_u\bar{\sigma}' dW_u\right)^2 |g_1 = x\right]
\end{aligned}$$

- Apply *formula 3.*

$$\begin{aligned}
& F(0)^2\mathbf{E}\left[\left(\int_0^T I_{\mu}(u)Y_u\bar{\sigma}' dW_u\right)^2 |g_1 = x\right] \\
= \;& F(0)^2\times\left[I_2^3\left(I_{\mu}Y\bar{\sigma}, I_{\mu}Y\bar{\sigma};T\right)\times\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma^2}\right) + I_0^3\left(I_{\mu}Y\bar{\sigma}, I_{\mu}Y\bar{\sigma};T\right)\right] \\
=: \;& d_4\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma^2}\right) + e_4
\end{aligned}$$

- Apply *formula 6.*

$$\begin{aligned}
& 2F(0)^2\mathbf{E}\left[\left(\int_0^T I_{\mu}(u)Y_u\bar{\sigma}' dW_u\right)\left(\int_0^T\left(\int_0^u Y_s^{-1}\omega'(s)dW_s\right)Y_u\bar{\sigma}' dW_u\right)|g_1 = x\right] \\
= \;& 2F(0)^2\times\left[I_3^6\left(Y^{-1}\omega, Y\bar{\sigma}, I_{\mu}Y\bar{\sigma};T\right)\times\left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}\right) + I_1^6\left(Y^{-1}\omega, Y\bar{\sigma}, I_{\mu}Y\bar{\sigma};T\right)\times\frac{x}{\Sigma}\right] \\
=: \;& f_4\left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}\right) + h_4\frac{x}{\Sigma}
\end{aligned}$$

- Apply *formula 7*.

$$
\begin{aligned}
& F(0)^2\mathbf{E}\left[\left(\int_0^T\left(\int_0^u Y_s^{-1}\omega'(s)dW_s\right)Y_u\bar{\sigma}'dW_u\right)^2|g_1=x\right] \\
= \quad & F(0)^2\times \\
& \Big[I_4^7(Y^{-1}\omega,Y\bar{\sigma},Y^{-1}\omega,Y\bar{\sigma};T)\times\left(\frac{x^4}{\Sigma^4}-\frac{6x^2}{\Sigma^3}+\frac{3}{\Sigma^2}\right) \\
& \qquad +I_2^7(Y^{-1}\omega,Y\bar{\sigma},Y^{-1}\omega,Y\bar{\sigma};T)\times\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma^2}\right)+I_0^7(Y^{-1}\omega,Y\bar{\sigma},Y^{-1}\omega,Y\bar{\sigma};T)\Big] \\
=: \quad & i_4\left(\frac{x^4}{\Sigma^4}-\frac{6x^2}{\Sigma^3}+\frac{3}{\Sigma^2}\right)+j_4\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma^2}\right)+k_4
\end{aligned}
$$

3. Apply *formula 7*.

$$
\begin{aligned}
& \frac{1}{F(0)^2}\mathbf{E}\left[\left(\int_0^T A_u^{(1)}\sigma_X(u)'dW_u\right)^2|g_1=x\right] \\
= \quad & \frac{1}{F(0)^2}\times \\
& \Big[I_4^7(\sigma_X,\sigma_X,\sigma_X,\sigma_X;T)\times\left(\frac{x^4}{\Sigma^4}-\frac{6x^2}{\Sigma^3}+\frac{3}{\Sigma^2}\right) \\
& \qquad +I_2^7(\sigma_X,\sigma_X,\sigma_X,\sigma_X;T)\times\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma^2}\right)+I_0^7(\sigma_X,\sigma_X,\sigma_X,\sigma_X;T)\Big] \\
=: \quad & l_4\left(\frac{x^4}{\Sigma^4}-\frac{6x^2}{\Sigma^3}+\frac{3}{\Sigma^2}\right)+m_4\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma^2}\right)+n_4
\end{aligned}
$$

4. Apply *formula 7*.

$$
\begin{aligned}
& 2F(0)^2\mathbf{E}\left[\left(\int_0^T g_{fi}^{(1)}(u)A_{fi}^{(1)}(u)dW_u\right)\left(\int_0^T g_{dj}^{(1)}(u)A_{dj}^{(1)}(u)dW_u\right)|g_1=x\right] \\
= \quad & 2F(0)^2\left(\frac{-\tau_i f_{fi}(0)}{(1+\tau_i f_{fi}(0))^2}\right)\left(\frac{-\tau_j f_{dj}(0)}{(1+\tau_j f_{dj}(0))^2}\right)\times \\
& \Big[I_4^7(\gamma_{fi},\gamma_{fi},\gamma_{dj},\gamma_{dj};T)\times\left(\frac{x^4}{\Sigma^4}-\frac{6x^2}{\Sigma^3}+\frac{3}{\Sigma^2}\right) \\
& +I_2^7(\gamma_{fi},\gamma_{fi},\gamma_{dj},\gamma_{dj};T)\times\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma^2}\right)+I_0^7(\gamma_{fi},\gamma_{fi},\gamma_{dj},\gamma_{dj};T)\Big] \\
=: \quad & o_{4i,j}\left(\frac{x^4}{\Sigma^4}-\frac{6x^2}{\Sigma^3}+\frac{3}{\Sigma^2}\right)+p_{4i,j}\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma^2}\right)+q_{4i,j}
\end{aligned}
$$

5.

$$
\begin{aligned}
& 2F(0)^2\mathbf{E}\left[\left(\int_0^T g_{di}^{(1)}(u)A_{di}^{(1)}(u)dW_u\right)\left(\int_0^T A_\sigma^{(1)}(u)\bar{\sigma}'dW_u\right)|g_1=x\right] \\
= \quad & 2F(0)^2\left(\frac{-\tau_i f_{di}(0)}{(1+\tau_i f_{di}(0))^2}\right)\mathbf{E}\left[\left(\int_0^T\gamma_{di}'(u)\left(\int_0^u\gamma_{di}'(s)dW_s\right)dW_u\right)\times\right. \\
& \left.\left\{\left(\int_0^T I_\mu(u)Y_u\bar{\sigma}'dW_u\right)+\left(\int_0^T\left(\int_0^u Y_s^{-1}\omega'(s)dW_s\right)Y_u\bar{\sigma}'dW_u\right)\right\}|g_1=x\right]
\end{aligned}
$$

- Apply *formula 6*.

$$
\begin{aligned}
& 2F(0)^2\left(\frac{-\tau_i f_{di}(0)}{(1+\tau_i f_{di}(0))^2}\right)\times \\
& \mathbf{E}\left[\left(\int_0^T\gamma_{di}'(u)\left(\int_0^u\gamma_{di}'(s)dW_s\right)dW_u\right)\left(\int_0^T I_\mu(u)Y_u\bar{\sigma}'dW_u\right)|g_1=x\right] \\
= \quad & 2F(0)^2\left(\frac{-\tau_i f_{di}(0)}{(1+\tau_i f_{di}(0))^2}\right)\times\left[I_3^6\left(\gamma_{di},\gamma_{di},I_\mu Y\bar{\sigma};T\right)\times\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)+I_1^6\left(\gamma_{di},\gamma_{di},I_\mu Y\bar{\sigma};T\right)\times\frac{x}{\Sigma}\right] \\
=: \quad & r_{4i}^d\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)+s_{4i}^d\frac{x}{\Sigma}
\end{aligned}
$$

- Apply *formula 7.*

$$\begin{aligned}
& 2F(0)^2\left(\frac{-\tau_i f_{di}(0)}{(1+\tau_i f_{di}(0))^2}\right)\times \\
& \mathbf{E}\left[\left(\int_0^T \gamma_{di}^{'}(u)\left(\int_0^u \gamma_{di}^{'}(s)dW_s\right)dW_u\right)\left(\int_0^T\left(\int_0^u Y_s^{-1}\omega^{'}(s)dW_s\right)Y_u\bar{\sigma}^{'}dW_u\right)|g_1=x\right] \\
= \quad & 2F(0)^2\left(\frac{-\tau_i f_{di}(0)}{(1+\tau_i f_{di}(0))^2}\right)\times \\
& \left[I_4^7(\gamma_{di},\gamma_{di},Y^{-1}\omega,Y\bar{\sigma};T)\times\left(\frac{x^4}{\Sigma^4}-\frac{6x^2}{\Sigma^3}+\frac{3}{\Sigma^2}\right)\right. \\
& \qquad \left.+I_2^7(\gamma_{di},\gamma_{di},Y^{-1}\omega,Y\bar{\sigma};T)\times\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma^2}\right)+I_0^7(\gamma_{di},\gamma_{di},Y^{-1}\omega,Y\bar{\sigma};T)\right] \\
=: \quad & t_{4i}^d\left(\frac{x^4}{\Sigma^4}-\frac{6x^2}{\Sigma^3}+\frac{3}{\Sigma^2}\right)+u_{4i}^d\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma^2}\right)+v_{4i}^d
\end{aligned}$$

6. Apply *formula 7.*

$$\begin{aligned}
& 2\times\mathbf{E}\left[\left(\int_0^T g_{di}^{(1)}(u)A_{di}^{(1)}(u)dW_u\right)\left(\int_0^T A_u^{(1)}\sigma_X(u)^{'}dW_u\right)|g_1=x\right] \\
= \quad & 2\left(\frac{-\tau_i f_{di}(0)}{(1+\tau_i f_{di}(0))^2}\right)\times \\
& \left[I_4^7(\gamma_{di},\gamma_{di},\sigma_X,\sigma_X;T)\times\left(\frac{x^4}{\Sigma^4}-\frac{6x^2}{\Sigma^3}+\frac{3}{\Sigma^2}\right)\right. \\
& \qquad \left.+I_2^7(\gamma_{di},\gamma_{di},\sigma_X,\sigma_X;T)\times\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma^2}\right)+I_0^7(\gamma_{di},\gamma_{di},\sigma_X,\sigma_X;T)\right] \\
=: \quad & w_{4i}^d\left(\frac{x^4}{\Sigma^4}-\frac{6x^2}{\Sigma^3}+\frac{3}{\Sigma^2}\right)+x_{4i}^d\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma^2}\right)+y_{4i}^d
\end{aligned}$$

7.

$$\begin{aligned}
& 2\left(\int_0^T A_\sigma^{(1)}(u)\bar{\sigma}^{'}dW_u\right)\left(\int_0^T A_u^{(1)}\sigma_X(u)^{'}dW_u\right) \\
= \quad & 2\times\mathbf{E}\left[\left(\int_0^T\left(\int_0^u \sigma_X(s)^{'}dW_s\right)\sigma_X(u)^{'}dW_u\right)\times\right. \\
& \left.\left\{\left(\int_0^T I_\mu(u)Y_u\bar{\sigma}^{'}dW_u\right)+\left(\int_0^T\left(\int_0^u Y_s^{-1}\omega^{'}(s)dW_s\right)Y_u\bar{\sigma}^{'}dW_u\right)\right\}|g_1=x\right]
\end{aligned}$$

- Apply *formula 6.*

$$\begin{aligned}
& 2\times\mathbf{E}\left[\left(\int_0^T \sigma_X^{'}(u)\left(\int_0^u \sigma_X^{'}(s)dW_s\right)dW_u\right)\left(\int_0^T I_\mu(u)Y_u\bar{\sigma}^{'}dW_u\right)|g_1=x\right] \\
= \quad & 2\times\left[I_3^6\left(\sigma_X,\sigma_X,I_\mu Y\bar{\sigma};T\right)\times\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)+I_1^6\left(\sigma_X,\sigma_X,I_\mu Y\bar{\sigma};T\right)\times\frac{x}{\Sigma}\right] \\
=: \quad & z_4\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)+aa_4\frac{x}{\Sigma}
\end{aligned}$$

- Apply *formula 7.*

$$\begin{aligned}
& 2\times\mathbf{E}\left[\left(\int_0^T \sigma_X^{'}(u)\left(\int_0^u \sigma_X^{'}(s)dW_s\right)dW_u\right)\left(\int_0^T\left(\int_0^u Y_s^{-1}\omega^{'}(s)dW_s\right)Y_u\bar{\sigma}^{'}dW_u\right)|g_1=x\right] \\
= \quad & 2\times\left[I_4^7(\sigma_X,\sigma_X,Y^{-1}\omega,Y\bar{\sigma};T)\times\left(\frac{x^4}{\Sigma^4}-\frac{6x^2}{\Sigma^3}+\frac{3}{\Sigma^2}\right)\right. \\
& \qquad \left.+I_2^7(\sigma_X,\sigma_X,Y^{-1}\omega,Y\bar{\sigma};T)\times\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma^2}\right)+I_0^7(\sigma_X,\sigma_X,Y^{-1}\omega,Y\bar{\sigma};T)\right] \\
=: \quad & ab_4\left(\frac{x^4}{\Sigma^4}-\frac{6x^2}{\Sigma^3}+\frac{3}{\Sigma^2}\right)+ac_4\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma^2}\right)+ad_4
\end{aligned}$$

Consequently, $C_{4,0}, C_{4,1}, C_{4,2}, C_{4,3}$, and $C_{4,4}$ are defined as;

$$
\begin{aligned}
C_{4,0} &= \left[\sum_{i,j\in\hat{J}_{N+1}} (c^f_{4i,j} + c^d_{4i,j})\right] + e_4 + k_4 + n_4 - \left[\sum_{i,j\in\hat{J}_{N+1}} q_{4i,j}\right] + \left[\sum_{i\in\hat{J}_{N+1}} \left\{(v^f_{4i} + y^f_{4i}) - (v^d_{4i} + y^d_{4i})\right\}\right] + ad_4 \\
C_{4,1} &= h_4 - \left[\sum_{i\in\hat{J}_{N+1}} \left(s^f_{4i} - s^d_{4i}\right)\right] + aa_4 \\
C_{4,2} &= \left[\sum_{i,j\in\hat{J}_{N+1}} \left(b^f_{4i,j} + b^d_{4i,j}\right)\right] + d_4 + j_4 + m_4 - \left[\sum_{i,j\in\hat{J}_{N+1}} p_{4i,j}\right] + \left[\sum_{i\in\hat{J}_{N+1}} \left\{(u^f_{4i} + x^f_{4i}) - (u^d_{4i} + x^d_{4i})\right\}\right] + ac_4 \\
C_{4,3} &= f_4 - \left[\sum_{i\in\hat{J}_{N+1}} \left(r^f_{4i} - r^d_{4i}\right)\right] + z_4 \\
C_{4,4} &= \left[\sum_{i,j\in\hat{J}_{N+1}} \left(a^f_{4i,j} + a^d_{4i,j}\right)\right] + i_4 + l_4 - \left[\sum_{i,j\in\hat{J}_{N+1}} o_{4i,j}\right] + \left[\sum_{i\in\hat{J}_{N+1}} \left\{(t^f_{4i} + w^f_{4i}) - (t^d_{4i} + w^d_{4i})\right\}\right] + ab_4.
\end{aligned}
$$

B Formulas

In this section, the formulas 1.- 7. used in the previous sections are listed up for convenience. They are derived by direct calculations using Gaussianity of the processes involved, which are straightforward, but lengthy and hence omitted. $W = \{(W^1_t, \cdots, W^d_t) : 0 \leq t\}$ denotes a d-dimensional Brownian motion. Let $q_i : [0, T] \mapsto \mathbf{R}^d, i = 1, 2, 3, 4, 5$ be non-random functions and define Σ as

$$\Sigma = \int_0^T q'_{1v} q_{1v} dv,$$

where z' is the transpose of z. We assume that $0 < \Sigma < \infty$ and integrability in the following formulas.

1.
$$\mathbf{E}\left[\int_0^T q'_{2t} dW_t | \int_0^T q'_{1v} dW_v = x\right] = \left(\int_0^T q'_{2t} q_{1t} dt\right) \frac{x}{\Sigma}$$

2.
$$
\begin{aligned}
&\mathbf{E}\left[\int_0^T \int_0^t q'_{2u} dW_u q'_{3t} dW_t | \int_0^T q'_{1v} dW_v = x\right] = \\
&\left(\int_0^T \int_0^t q'_{2u} q_{1u} du q'_{3t} q_{1t} dt\right) \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right)
\end{aligned}
$$

3.
$$
\begin{aligned}
&\mathbf{E}\left[\left(\int_0^T q'_{2u} dW_u\right) \left(\int_0^T q'_{3s} dW_s\right) | \int_0^T q'_{1v} dW_v = x\right] = \\
&\left(\int_0^T q'_{2u} q_{1u} du\right) \left(\int_0^T q'_{3s} q_{1s} ds\right) \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right) \\
&+ \int_0^T q'_{2t} q_{3t} dt
\end{aligned}
$$

4.
$$
\begin{aligned}
&\mathbf{E}\left[\int_0^T \int_0^t \int_0^s q'_{2u} dW_u q'_{3s} dW_s q'_{4t} dW_t | \int_0^T q'_{1v} dW_v = x\right] = \\
&\left(\int_0^T q'_{4t} q_{1t} \int_0^t q'_{3s} q_{1s} \int_0^s q'_{2u} q_{1u} du ds dt\right) \left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}\right)
\end{aligned}
$$

5.

$$
\mathbf{E}\left[\int_0^T \left(\int_0^t q_{2u}^{'} dW_u\right)\left(\int_0^t q_{3s}^{'} dW_s\right) q_{4t}^{'} dW_t | \int_0^T q_{1v}^{'} dW_v = x\right] =
$$

$$
\left\{\int_0^T \left(\int_0^t q_{2u}^{'} q_{1u} du\right)\left(\int_0^t q_{3s}^{'} q_{1s} ds\right) q_{4t}^{'} q_{1t} dt\right\}\left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}\right)
$$

$$
+\left(\int_0^T \int_0^t q_{2u}^{'} q_{3u} du q_{4t}^{'} q_{1t} dt\right)\frac{x}{\Sigma}
$$

6.

$$
\mathbf{E}\left[\left(\int_0^T \int_0^t q_{2s}^{'} dW_s q_{3t}^{'} dW_t\right)\left(\int_0^T q_{4u}^{'} dW_u\right) | \int_0^T q_{1v}^{'} dW_v = x\right] =
$$

$$
\left(\int_0^T q_{3t}^{'} q_{1t} \int_0^t q_{2s}^{'} q_{1s} ds dt\right)\left(\int_0^T q_{4u}^{'} q_{1u} du\right)\left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}\right)
$$

$$
+\left(\int_0^T q_{3t}^{'} q_{1t} \int_0^t q_{2s}^{'} q_{4s} ds dt\right)\frac{x}{\Sigma} + \left(\int_0^T q_{3t}^{'} q_{4t} \int_0^t q_{2s}^{'} q_{1s} ds dt\right)\frac{x}{\Sigma}
$$

7.

$$
\mathbf{E}\left[\left(\int_0^T \int_0^t q_{2s}^{'} dW_s q_{3t}^{'} dW_t\right)\left(\int_0^T \int_0^r q_{4u}^{'} dW_u q_{5r}^{'} dW_r\right) | \int_0^T q_{1v}^{'} dW_v = x\right] =
$$

$$
\left(\int_0^T q_{3t}^{'} q_{1t} \int_0^t q_{2s}^{'} q_{1s} ds dt\right)\left(\int_0^T q_{5r}^{'} q_{1r} \int_0^r q_{4u}^{'} q_{1u} du dr\right)\left(\frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} + \frac{3}{\Sigma^2}\right)
$$

$$
+\left(\int_0^T q_{3t}^{'} q_{1t} \int_0^t q_{5r}^{'} q_{1r} \int_0^r q_{2u}^{'} q_{4u} du dr dt\right)\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right)
$$

$$
+\left(\int_0^T q_{5t}^{'} q_{1t} \int_0^t q_{3r}^{'} q_{1r} \int_0^r q_{2u}^{'} q_{4u} du dr dt\right)\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right)
$$

$$
+\left(\int_0^T q_{3t}^{'} q_{1t} \int_0^t q_{2r}^{'} q_{5r} \int_0^r q_{4u}^{'} q_{1u} du dr dt\right)\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right)
$$

$$
+\left\{\int_0^T q_{3t}^{'} q_{5t} \left(\int_0^t q_{2s}^{'} q_{1s} ds\right)\left(\int_0^t q_{4u}^{'} q_{1u} du\right) dt\right\}\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right)
$$

$$
+\left(\int_0^T q_{5r}^{'} q_{1r} \int_0^r q_{3u}^{'} q_{4u} \int_0^u q_{2s}^{'} q_{1s} ds du dr\right)\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right)
$$

$$
+\int_0^T \int_0^t q_{2u}^{'} q_{4u} du q_{3t}^{'} q_{5t} dt
$$

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We thank Professor Seisho Sato in The Insutitute of Statistical Mathematics and Mr. Akira Yamazaki in Mizuho-DL Financial Technology Co., Ltd. for their precious advices on numerical computations in the sections 4.1 and 4.2. We also appriciate Mizuho-DL Financial Technology Co., Ltd. for providing data used in the section 4.2.

References

[1] Amin K.I. and Bodurtha, J.N. [1995], "Discrete-time valuation of American options with stochastic interest rates," *Review of Financial Studies*, Vol. 8, pp. 193-234.

[2] Amin K.I. and Jarrow R. [1991], "Pricing Foreign-Currency Options Under Stochastic Interest-Rates," *Journal of International Money and Finance*, Vol. 10, pp. 310-329.

[3] Bichteler, K., Gravereaux, J.B. and Jacod, J[1987], "Malliavin Calculus for Processes with Jumps," Gordon and Breach Science Publishers.

[4] Black, F. [1976], "The Pricing of Commodity Contracts," *Journal of Financial Economics*, Vol. 3, pp. 167-179.

[5] Black, F and Scholes, M [1973], "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy*, Vol. 81(3), pp. 637-654.

[6] Brace, A., Gatarek, D. and Musiela, M. [1998], "A Multifactor Gauss Markov Implementation of Heath, Jarrow, and Morton," *Mathematical Finance*, Vol. 7, pp. 127-154.

[7] Dempster, M.A.H., and Hutton, J.P. [1997], "Numerical valuation of cross-currency swaps and swaptions," *Mathematics of Derivatives Securities*, pp. 473-503, Cambridge Univ. Press.

[8] Garman, M. and Kohlhagen, S. [1983], "Foreign currency options," *Journal of International Money and Finance*, Vol. 2, pp. 231-237.

[9] Grabbe, J. [1983], "The Pricing of Call and Put Options on Foreign-Exchange," *Journal of International Money and Finance*, Vol. 2, pp. 239-253.

[10] Heath, D., Jarrow, R. and Morton, A.[1992], "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation," *Econometrica*, Vol. 60, pp. 77-105.

[11] Heston, S.[1993], "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options," *The Review of Financial Studies*, Vol.6, pp. 327-343.

[12] Hilliard, J., Madura, J. and Tucker, A. [1991], "Currency Option Pricing with Stochastic Domestic and Foreign Interest-Rates," *Journal of Financial and Quantitative Analysis*, Vol. 26, pp. 139-151.

[13] Hull, J. C. and White, A. D. [1990], "Pricing Interest Rate Derivate Securities," *Review of Financial Studies*, Vol. 3, pp. 573-592.

[14] Hull, J. C. and White, A. D. [1994a],
"Numerical Procedures for Implementing Term Structure Models I: Single-factor Models," *Journal of Derivatives*, Vol. 2(Fall), pp. 7-16.

[15] Ikeda, N. and Watanabe, S. [1989], *Stochastic Differential Equations and Diffusion Processes*, Second Edition, North-Holland/Kodansha, Tokyo.

[16] Karatzas, I. Lehoczky, J. and Shreve, S. [1987], "Optimal Portfolio and Consumption Decisions for a 'small investor' on a Finite Horizon," *SIAM Journal of Control and Optimaization*, Vol. 25, pp. 1157-1186.

[17] Karatzas, I. and Shreve, S. [1998], *Methods of Mathematical Finance*, Springer.

[18] Kawai, A. [2003], "A New Approximate Swaption Formula in the LIBOR Market Model: An Asymptotic Expansion Approach," *Applied Mathematical Finance*, Vol.10, pp. 49-74.

[19] Kobayashi, T., Takahashi, A. and Tokioka, N. [2003], "Dynamic Optimality of Yield Curve Strategies, " *International Review of Finance*, Vol.4, pp. 49-78.

[20] Kunitomo, N. and Kim, Y-J.[2005], "Effects of Stochastic Interest Rates and Volatility on Contingent Claims," forthcoming *Japanese Economic Review.*

[21] Kunitomo, N. and Takahashi, A. [1992], "Pricing Average Options," *Japan Financial Review*, Vol.14, pp. 1-20. (In Japanese.)

[22] Kunitomo, N. and Takahashi, A. [2001], "The Aysmptotic Expansion Approach to the Valuation of Interest Rate Contingent Claims," *Mathematical Finance*, Vol.11, pp. 117-151.

[23] Kunitomo, N. and Takahashi, A. [2003], "On Validity of the Asymptotic Expansion Approach in Contingent Claim Analysis," *Annals of Applied Probability*, Vol.13(3)

[24] Kunitomo, N. and Takahashi, A. [2004], "Applications of the Asymptotic Expansion Approach based on Malliavin-Watanabe Calculus in Financial Problems," *Stochastic Processes and Applications to Mathematical Finance*, pp. 195-232, World Scientific.

[25] Lütkebohmert, E. [2004a], "Finite Dimensional Realizations of Interest Rate Models with Jumps and an Asymptotic Expansion for the Black-Scholes Model with Generalized Volatility," Unpublished Ph.D. Dissertation, Institute for Applied Mathematics, University of Bonn.

[26] Lütkebohmert, E. [2004b], "An Asymptotic Expansion for a Black-Scholes Type Model," Bull. Sci. math., Vol.128, pp.661-685.

[27] Malliavin, P. and Thalmaier, A. [2006], *Stochastic Calculus of Variations in Mathematical Finance*, Springer.

[28] Matsuoka, R., Takahashi, A. and Uchida, Y. [2006], "A New Computational Scheme for Computing Greeks by the Asymptotic Expansion Approach," *Asia-Pacific Financial Markets*, Vol.11, pp. 393-430.

[29] Melino, A. and Turnbull, S. [1991], “The Pricing of Foreign-Currency Options,” *Canadian Journal of Economics*, Vol. 24, pp. 251-281.

[30] Merton, R.C. [1973], “Theory of Rational Option Pricing”, *Bell Journal of Economics and Management Science*, Vol.4(1), pp. 141-183.

[31] Mikkelsen, P. [2001], “Cross-currency LIBOR market models,” Working paper, University of Aarhus.

[32] Miltersen, K., Sandmann, K. and Sondermann, D. [1997], “Closed Form Solutions for Term Structure Derivatives with Long-normal Interest Rates,” *Journal of Finance*, Vol. 52, pp. 409-430.

[33] Muroi, Y. [2005], “Pricing Contingent Claims with Credit Risk: Asymptotic Expansion Approach,” *Finance and Stochastics*, Vol 9(3), pp. 415-427.

[34] Piterbarg, V. [2005], “A multi-currency model with FX volatility skew,” Working paper.

[35] Rebonato, R.[2004], “Volatility and Correlation,” pp. 427-436, John Wiley & Sons, Ltd.

[36] Rumsey, J. [1991], “Pricing Cross-Currency Options,” *Journal of Futures Markets*, Vol. 11, pp. 89-93.

[37] Schlögl, E.[2002], “A Multi-currency Extension of the Lognormal Interest Rate Market Model,” *Finance and Stochastics*, pp. 173-196.

[38] Takahashi, A. [1995], “Essays on the Valuation Problems of Contingent Claims,” Unpublished Ph.D. Dissertation, Haas School of Business, University of California, Berkeley.

[39] Takahashi, A. [1999], “An Asymptotic Expansion Approach to Pricing Contingent Claims,” *Asia-Pacific Financial Markets*, Vol.6, pp. 115-151.

[40] Takahashi, A. [2005], “A Note on Computing Greeks by an Asymptotic Expansion Scheme,” Preprint.

[41] Takahashi, A. and Saito, T.[2003], “An Asymptotic Expansion Approach to Pricing American Options,” *Monetary and Economic Studies*, Vol. 22, pp. 35-87, Bank of Japan. (In Japanese.)

[42] Takahashi, A. and Takehara, K.[2008], “Fourier Transform Method with an Asymptotic Expansion Approach: an Application to Currency Options,” *International Journal of Theoretical and Applied Finance*, Vol. 11(4), pp. 381-401.

[43] Takahashi, A. and Tokioka, N. [1999], “Valuation of Cross-currency Derivatives with Stochastic Interest Rates in a Three-factor Model,” *Gendai Finance*, Vol. 5, pp. 3-16. (In Japanese.)

[44] Takahashi, A. and Uchida, Y. [2006], “New Acceleration Schemes with the Asymptotic Expansion in Monte Carlo Simulation, ”*Advances in Mathematical Economics*, Vol. 8, pp. 411-431.

[45] Takahashi, A. and Yoshida, N.[2004], “An Asymptotic Expansion Scheme for Optimal Investment Problems,” *Statistical Inference for Stochastic Processes*, Vol.7(2), pp. 153-188.

[46] Takahashi, A. and Yoshida, N. [2005], “Monte Carlo Simulation with Asymptotic Method,” *Journal of Japan Statistical Society*, Vol. 35(2), pp. 171-203.

[47] Watanabe, S. [1987], “Analysis of Wiener Functionals (Malliavin Calculus) and Its Applications to Heat Kernels,” *The Annals of Probability*, Vol. 15, pp. 1-39.

[48] Yoshida, N. [1992a], ”Asymptotic Expansion for Small Diffusions via the Theory of Malliavin-Watanabe,” *Probability Theory and Related Fields*, Vol. 92, pp. 275-311.

[49] Yoshida, N. [1992b], “Asymptotic Expansions for Statistics Related to Small Diffusions,” *Journal of Japan Statistical Society*, Vol.22, pp. 139-159.

Table 2:Largest differences/relative differences of the second order approximations.

	1y		2y		3y		4y		5y	
	diff.	relative diff.	diff.	relative diff.	diff.	relative diff.	diff.	relative diff.	diff.	relative diff.
Corr.1	−0.057	−5.76%	−0.149	−10.50%	−0.258	−15.28%	−0.354	−13.79%	−0.502	−18.13%
Corr.2	−0.044	−5.70%	−0.101	−9.80%	−0.173	−15.21%	−0.242	−12.83%	−0.357	−18.05%
Corr.3	−0.056	−5.29%	−0.138	−8.78%	−0.229	−10.57%	−0.333	−12.82%	−0.444	−13.97%

Table 3:Largest relative differences and their levels of the second order approximations.

	1y		2y		3y		4y		5y	
	relative diff.		relative diff.		relative diff.		relative diff.		relative diff.	
Corr.1	−5.76%	−0.057	−10.50%	−0.149	−15.28%	−0.258	−13.79%	−0.354	−18.13%	−0.502
Corr.2	−5.70%	−0.044	−9.80%	−0.101	−15.21%	−0.173	−12.83%	−0.242	−18.05%	−0.357
Corr.3	−5.29%	−0.056	−8.78%	−0.138	−10.57%	−0.226	−12.82%	−0.333	−13.97%	−0.427

Table 4:Largest differences/relative differences of the third order approximations.

	1y		2y		3y		4y		5y	
	diff.	relative diff.	diff.	relative diff.	diff.	relative diff.	diff.	relative diff.	diff.	relative diff.
Corr.1	0.009	0.45%	0.026	0.49%	0.052	0.39%	0.076	0.83%	0.133	0.86%
Corr.2	0.020	−2.76%	−0.038	−3.69%	0.078	−5.26%	0.115	−3.60%	0.177	−4.57%
Corr.3	0.008	0.22%	0.022	1.11%	0.050	0.78%	0.080	1.59%	0.124	1.60%

Table 5:Largest relative differences and their levels of the third order approximations.

	1y		2y		3y		4y		5y	
	relative diff.		relative diff.		relative diff.		relative diff.		relative diff.	
Corr.1	0.45%	0.004	0.49%	0.007	0.39%	0.007	0.83%	0.019	0.86%	0.133
Corr.2	−2.76%	−0.022	−3.65%	−0.038	−5.26%	−0.060	−3.60%	−0.059	−4.57%	−0.078
Corr.3	0.22%	0.002	1.11%	0.017	0.78%	0.016	1.59%	0.038	1.60%	0.045

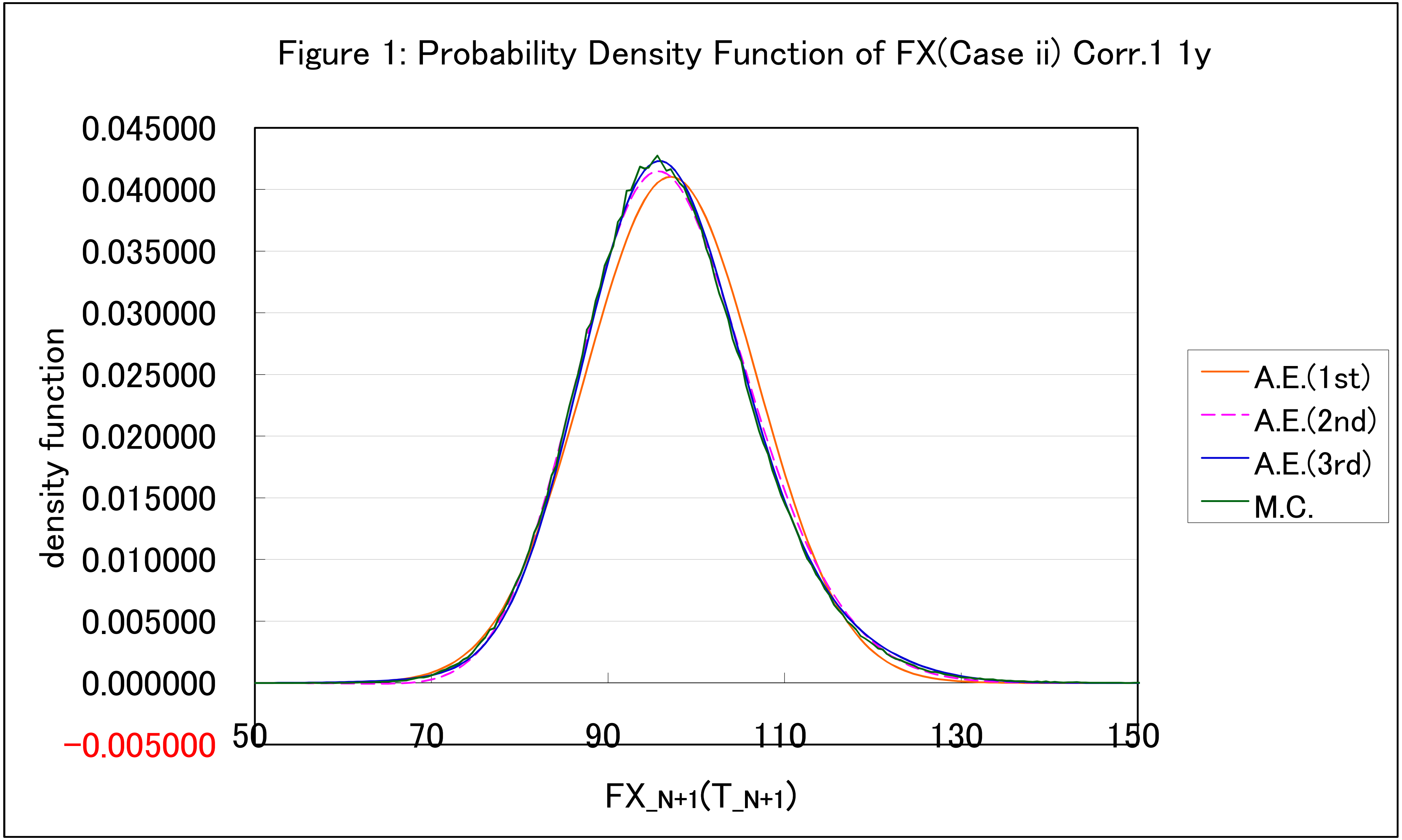

Figure 1: Probability Density Function of FX(Case ii) Corr.1 1y
density function
0.045000
0.040000
0.035000
0.030000
0.025000
0.020000
0.015000
0.010000
0.005000
0.000000
-0.005000
50
70
90
110
130
150
FX_N+1(T_N+1)
A.E.(1st)
A.E.(2nd)
A.E.(3rd)
M.C.

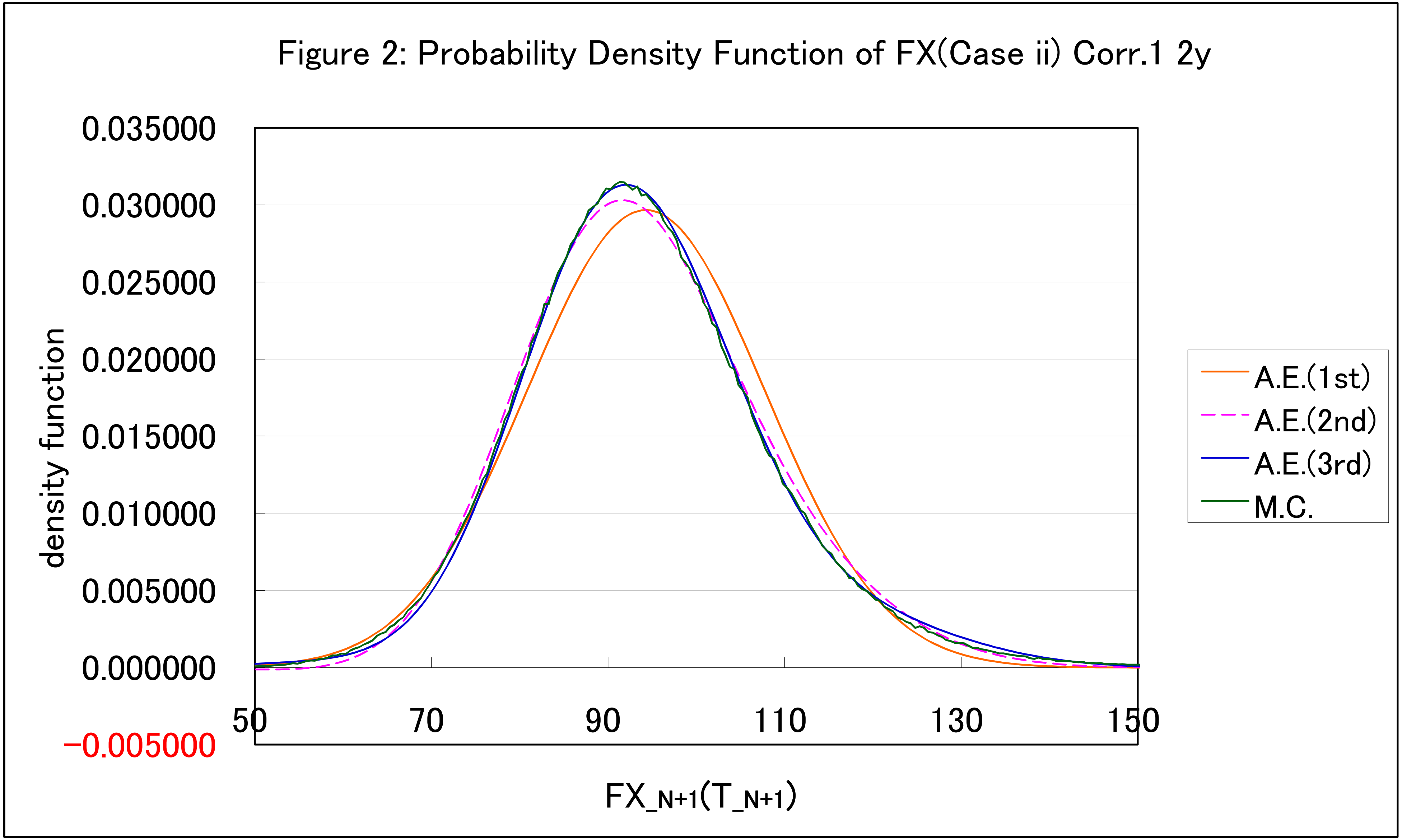
Figure 2: Probability Density Function of FX(Case ii) Corr.1 2y
0.035000
0.030000
0.025000
0.020000
0.015000
0.010000
0.005000
0.000000
−0.005000
density function
50
70
90
110
130
150
FX_N+1(T_N+1)
A.E.(1st)
A.E.(2nd)
A.E.(3rd)
M.C.

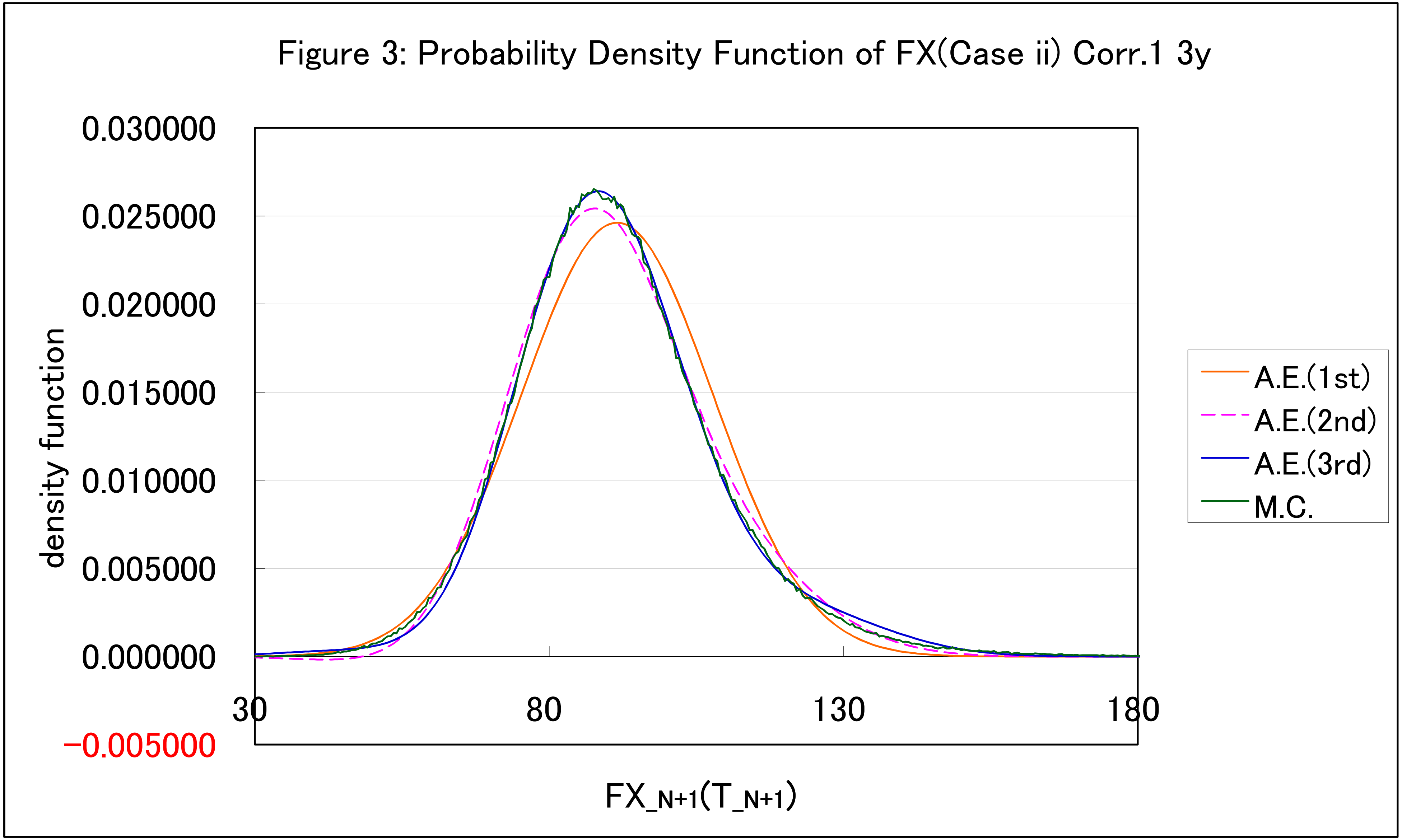

Figure 3: Probability Density Function of FX(Case ii) Corr.1 3y
density function
0.030000
0.025000
0.020000
0.015000
0.010000
0.005000
0.000000
−0.005000
30
80
130
180
FX_N+1(T_N+1)
A.E.(1st)
A.E.(2nd)
A.E.(3rd)
M.C.

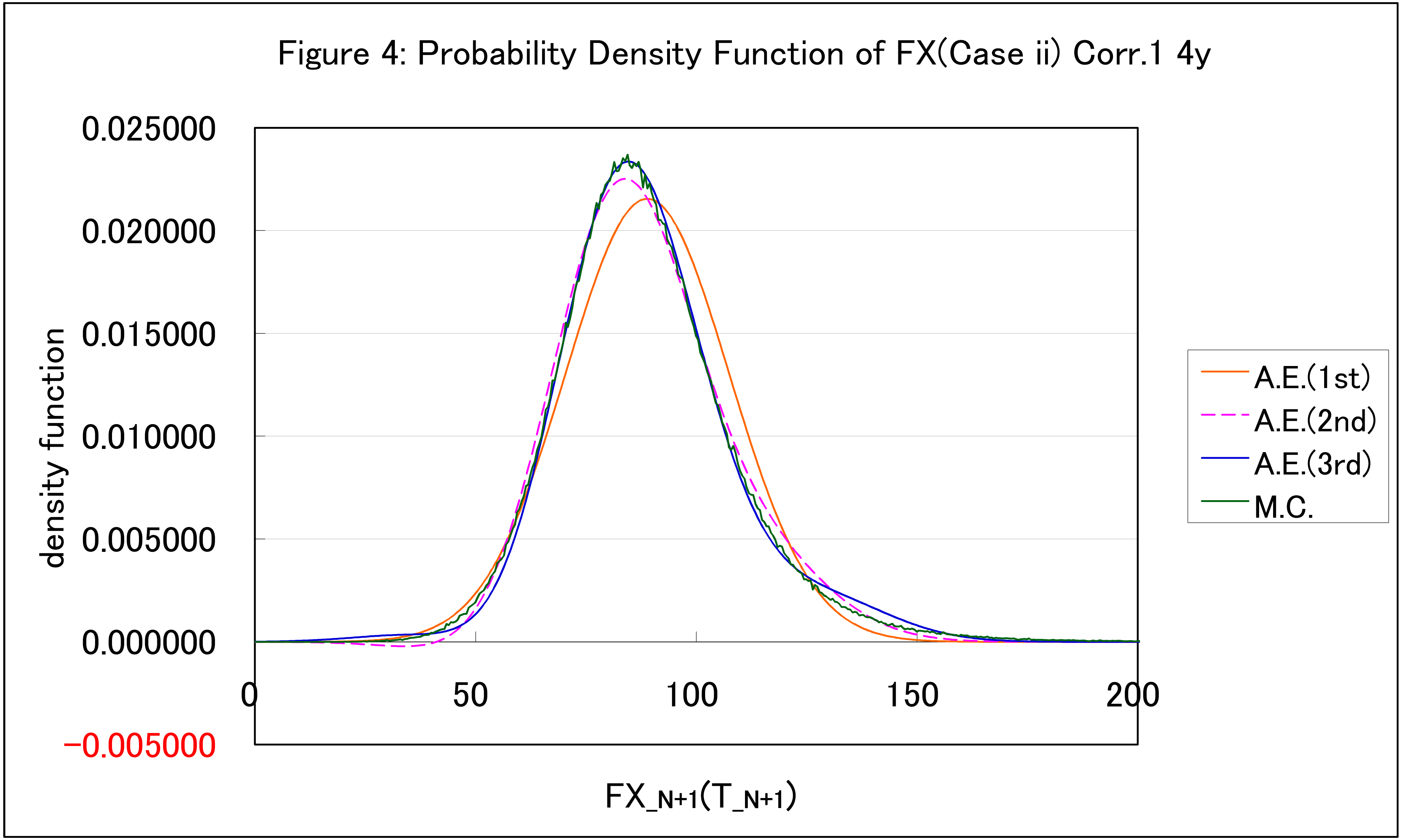
Figure 4: Probability Density Function of FX(Case ii) Corr.1 4y
density function
0.025000
0.020000
0.015000
0.010000
0.005000
0.000000
−0.005000
0
50
100
150
200
FX_N+1(T_N+1)
A.E.(1st)
A.E.(2nd)
A.E.(3rd)
M.C.

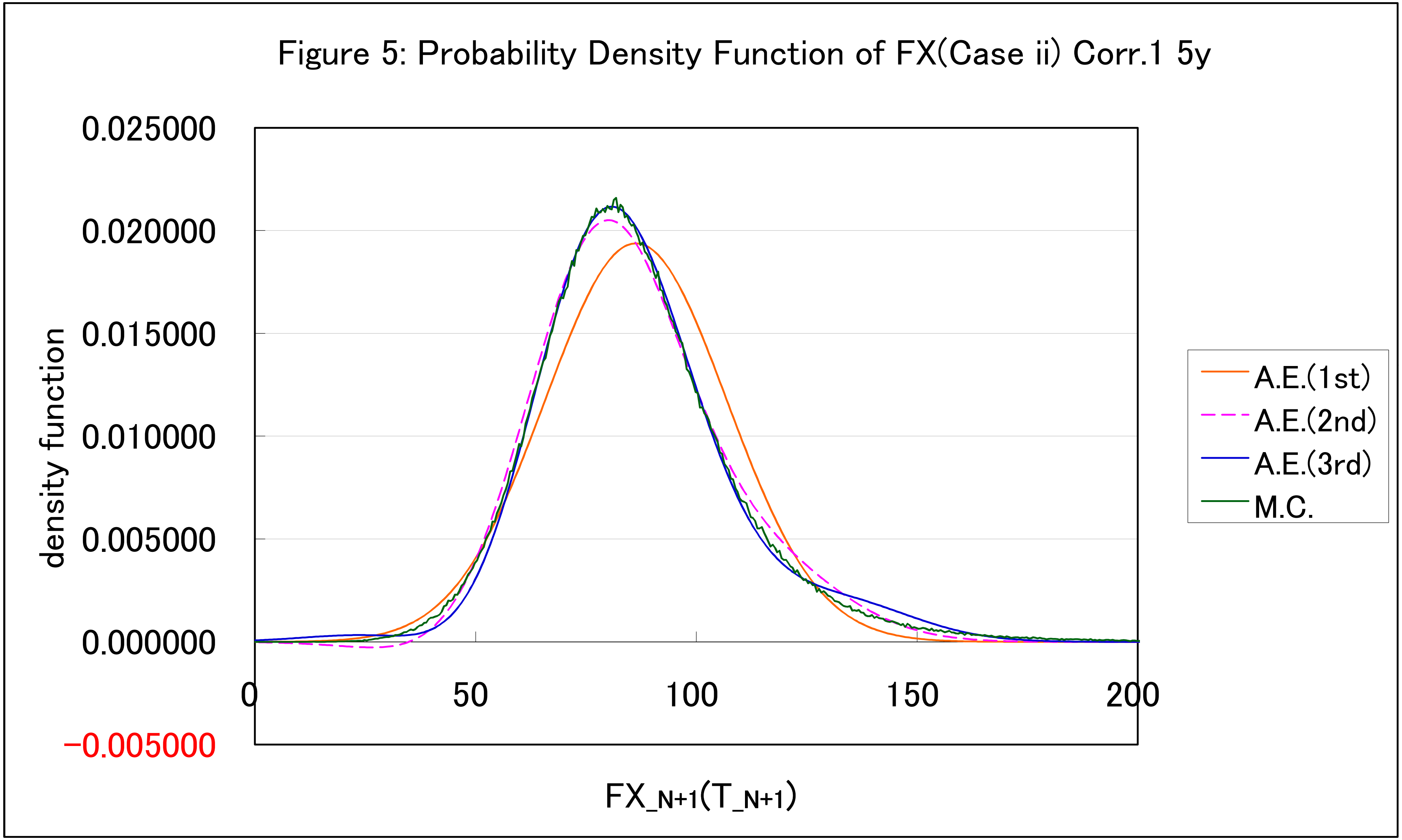
Figure 5: Probability Density Function of FX(Case ii) Corr.1 5y
0.025000
0.020000
0.015000
0.010000
0.005000
0.000000
-0.005000
density function
0
50
100
150
200
FX_N+1(T_N+1)
A.E.(1st)
A.E.(2nd)
A.E.(3rd)
M.C.

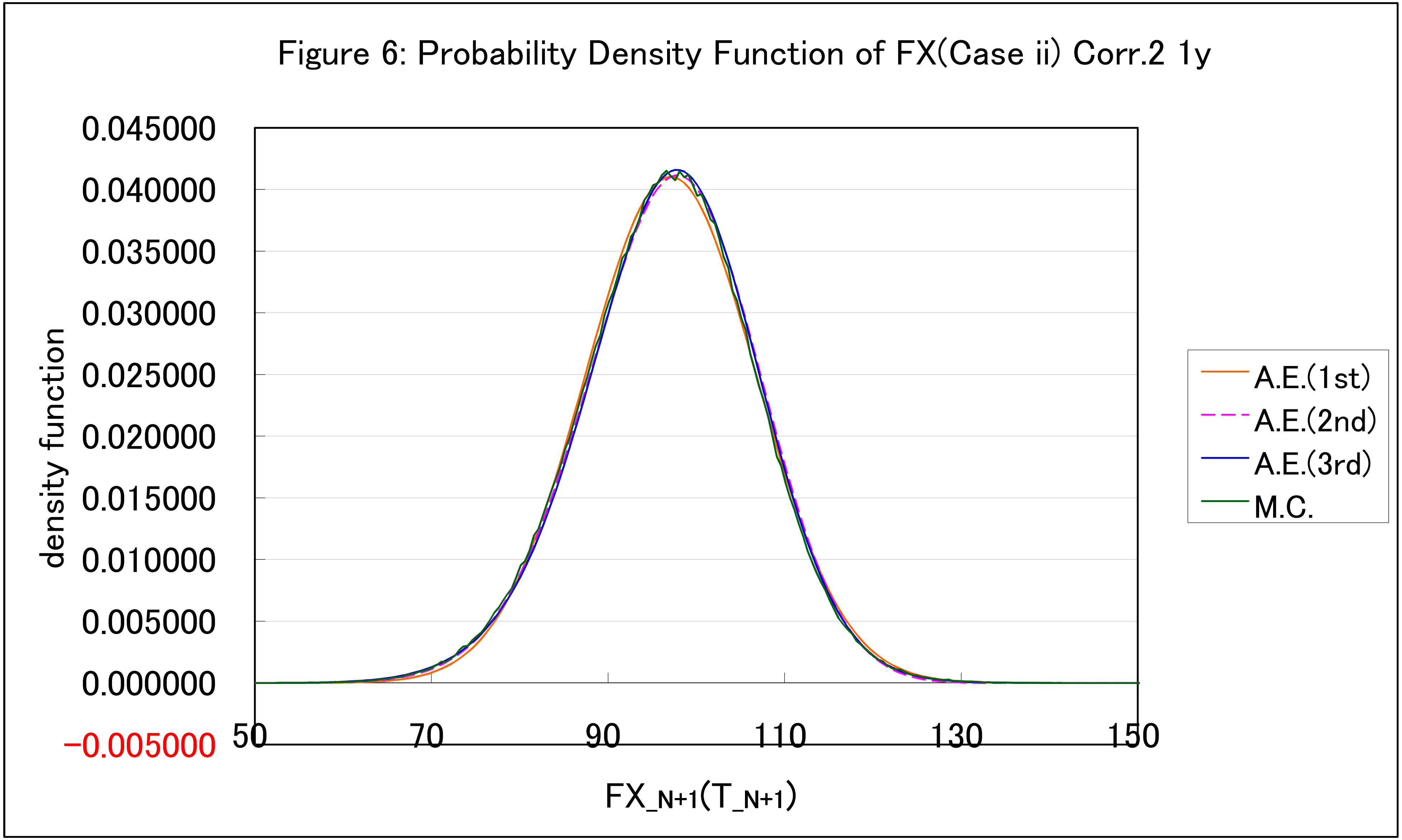

Figure 6: Probability Density Function of FX(Case ii) Corr.2 1y
density function
0.045000
0.040000
0.035000
0.030000
0.025000
0.020000
0.015000
0.010000
0.005000
0.000000
-0.005000
50
70
90
110
130
150
FX_N+1(T_N+1)
A.E.(1st)
A.E.(2nd)
A.E.(3rd)
M.C.

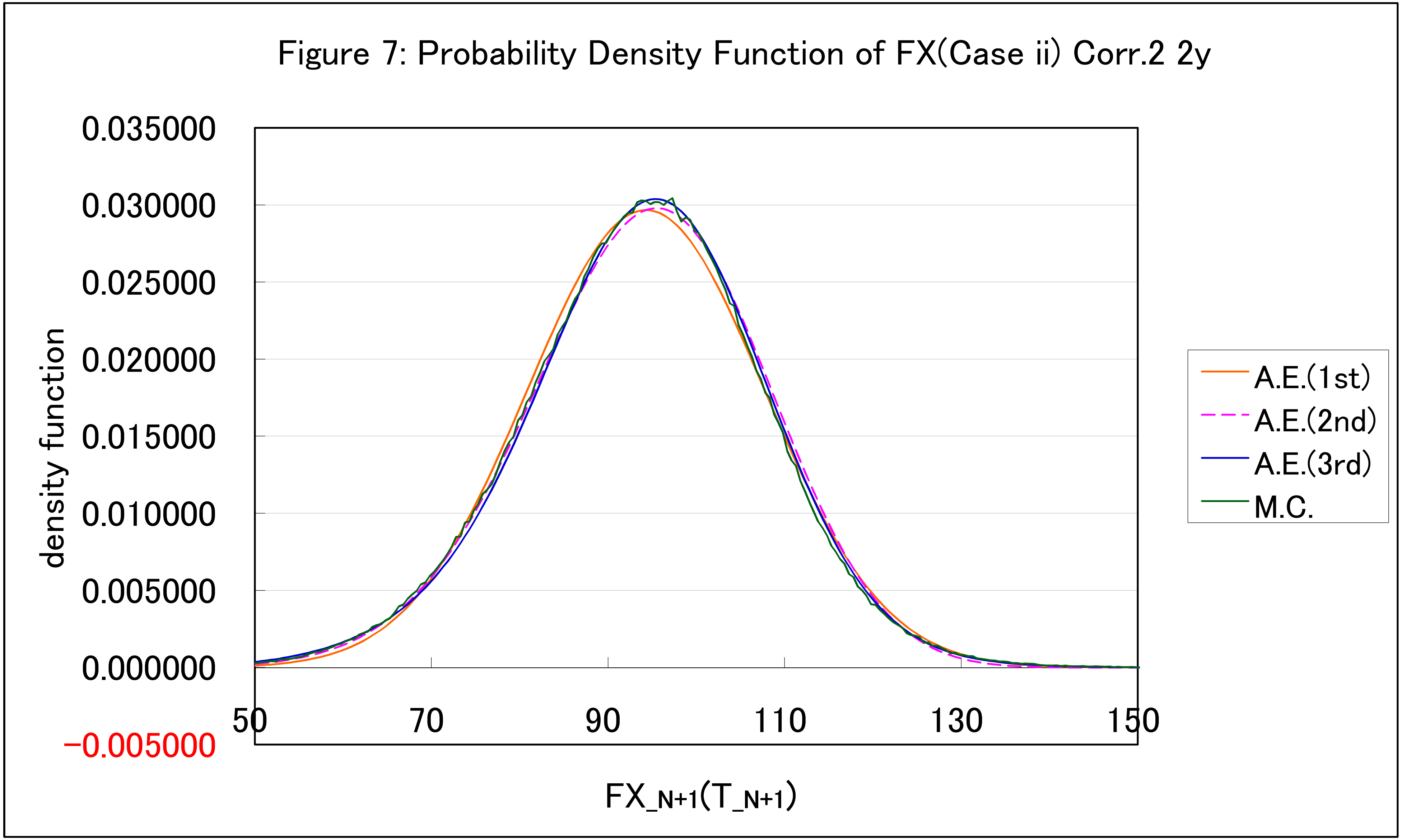
Figure 7: Probability Density Function of FX(Case ii) Corr.2 2y
0.035000
0.030000
0.025000
0.020000
0.015000
0.010000
0.005000
0.000000
−0.005000
density function
50
70
90
110
130
150
FX_N+1(T_N+1)
A.E.(1st)
A.E.(2nd)
A.E.(3rd)
M.C.

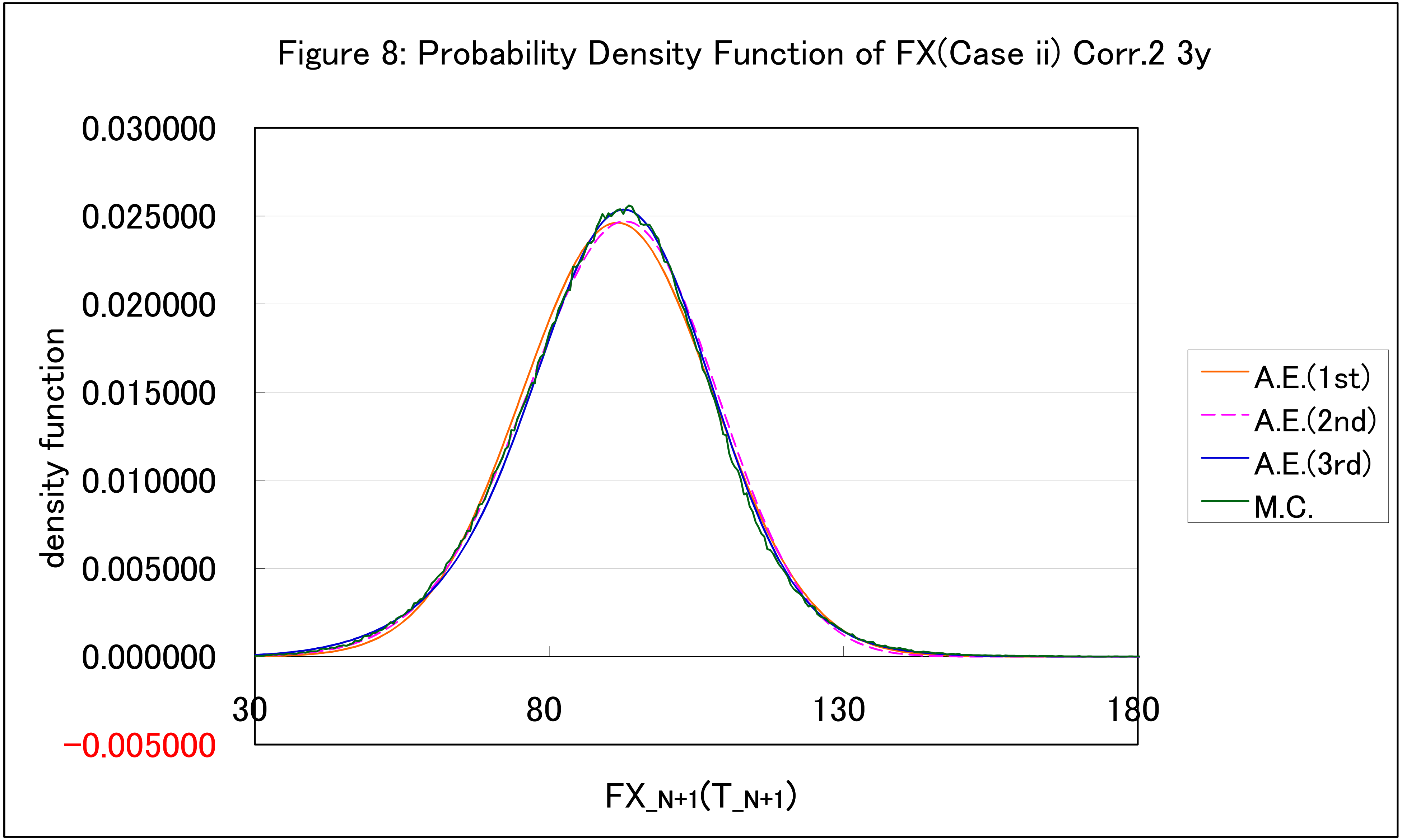
Figure 8: Probability Density Function of FX(Case ii) Corr.2 3y
density function
0.030000
0.025000
0.020000
0.015000
0.010000
0.005000
0.000000
−0.005000
30
80
130
180
FX_N+1(T_N+1)
A.E.(1st)
A.E.(2nd)
A.E.(3rd)
M.C.

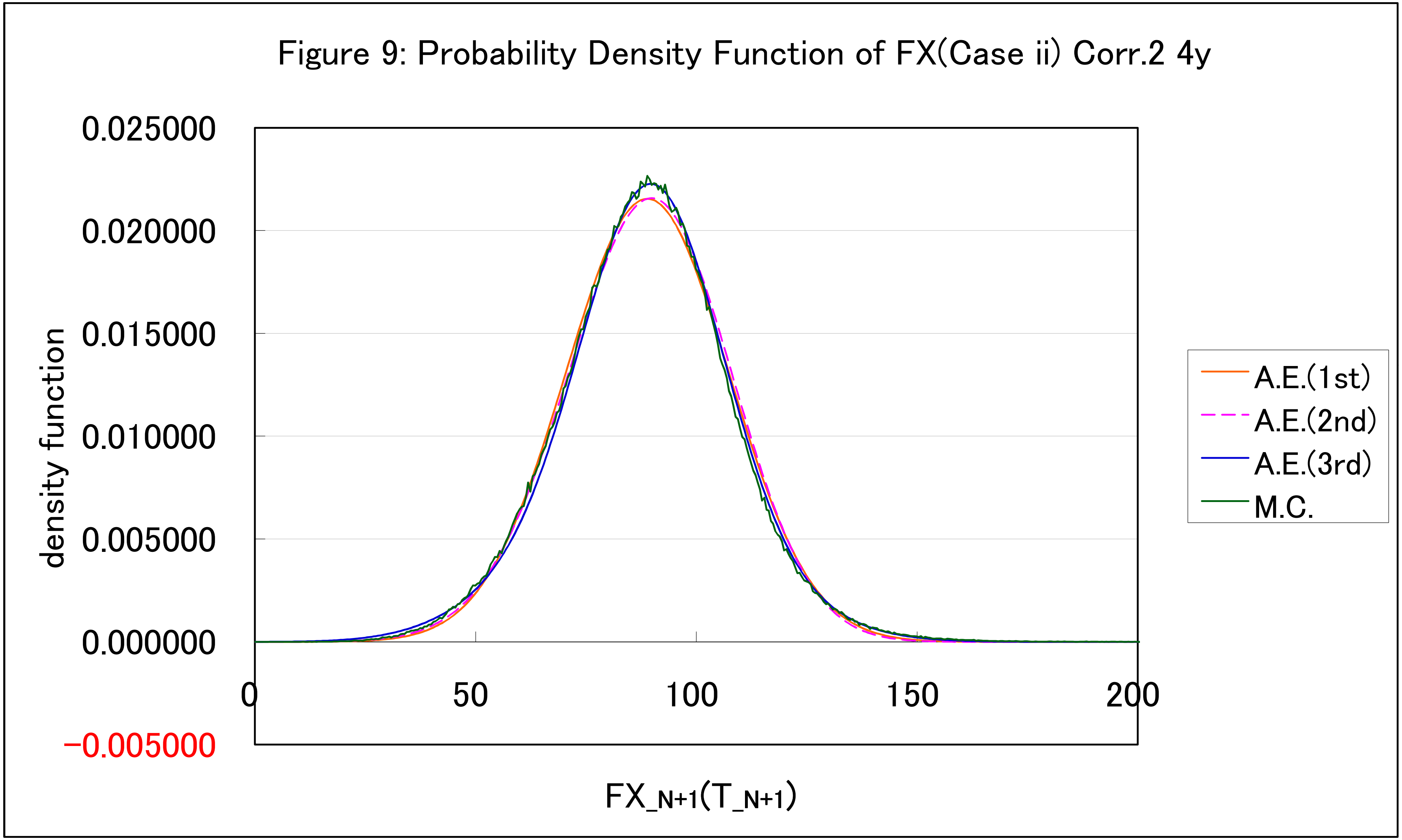

Figure 9: Probability Density Function of FX(Case ii) Corr.2 4y
density function
0.025000
0.020000
0.015000
0.010000
0.005000
0.000000
−0.005000
0
50
100
150
200
FX_N+1(T_N+1)
A.E.(1st)
A.E.(2nd)
A.E.(3rd)
M.C.

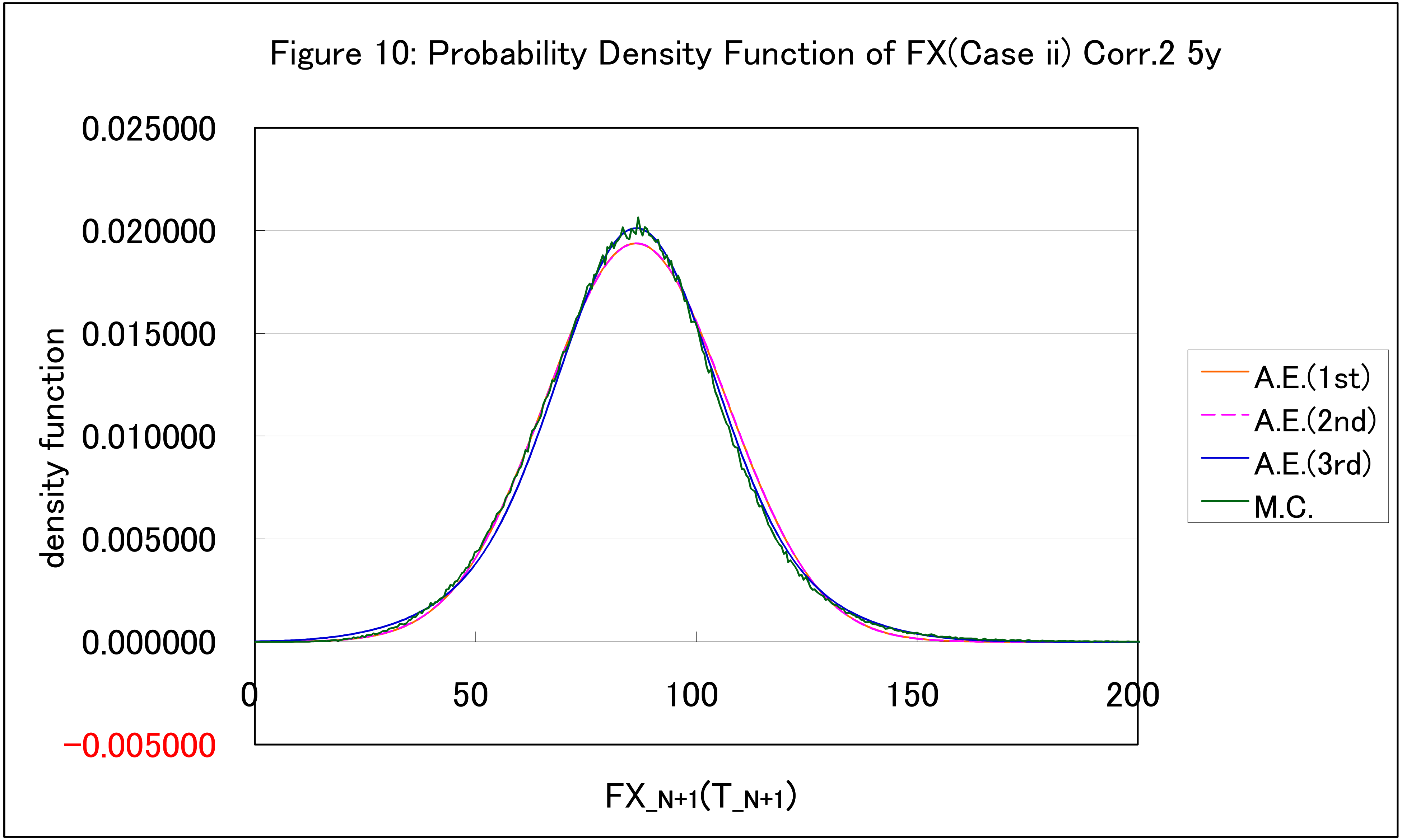

Figure 10: Probability Density Function of FX(Case ii) Corr.2 5y
0.025000
0.020000
0.015000
0.010000
0.005000
0.000000
-0.005000
0
50
100
150
200
density function
FX_N+1(T_N+1)
A.E.(1st)
A.E.(2nd)
A.E.(3rd)
M.C.

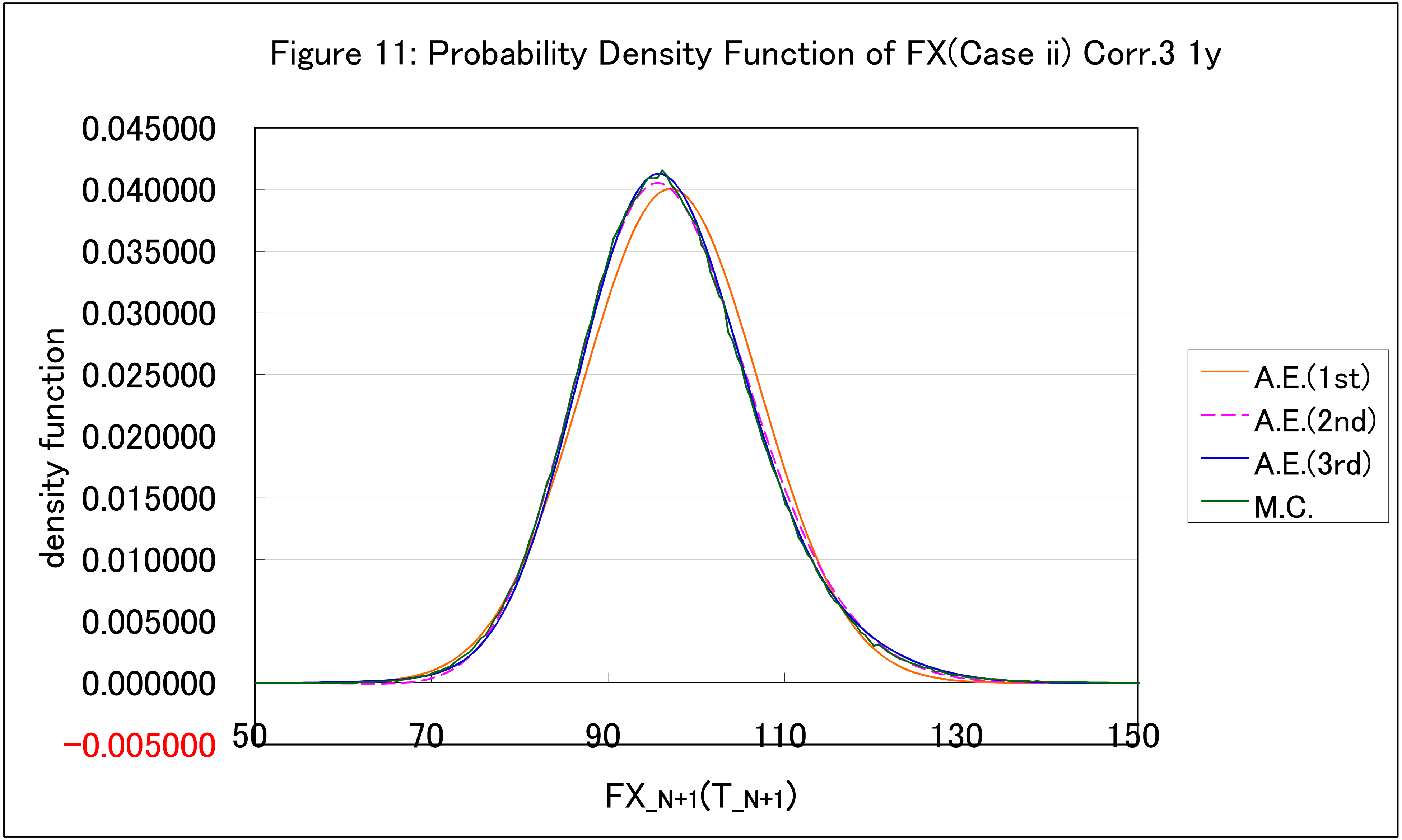
Figure 11: Probability Density Function of FX(Case ii) Corr.3 1y
density function
0.045000
0.040000
0.035000
0.030000
0.025000
0.020000
0.015000
0.010000
0.005000
0.000000
-0.005000
50
70
90
110
130
150
FX_N+1(T_N+1)
A.E.(1st)
A.E.(2nd)
A.E.(3rd)
M.C.

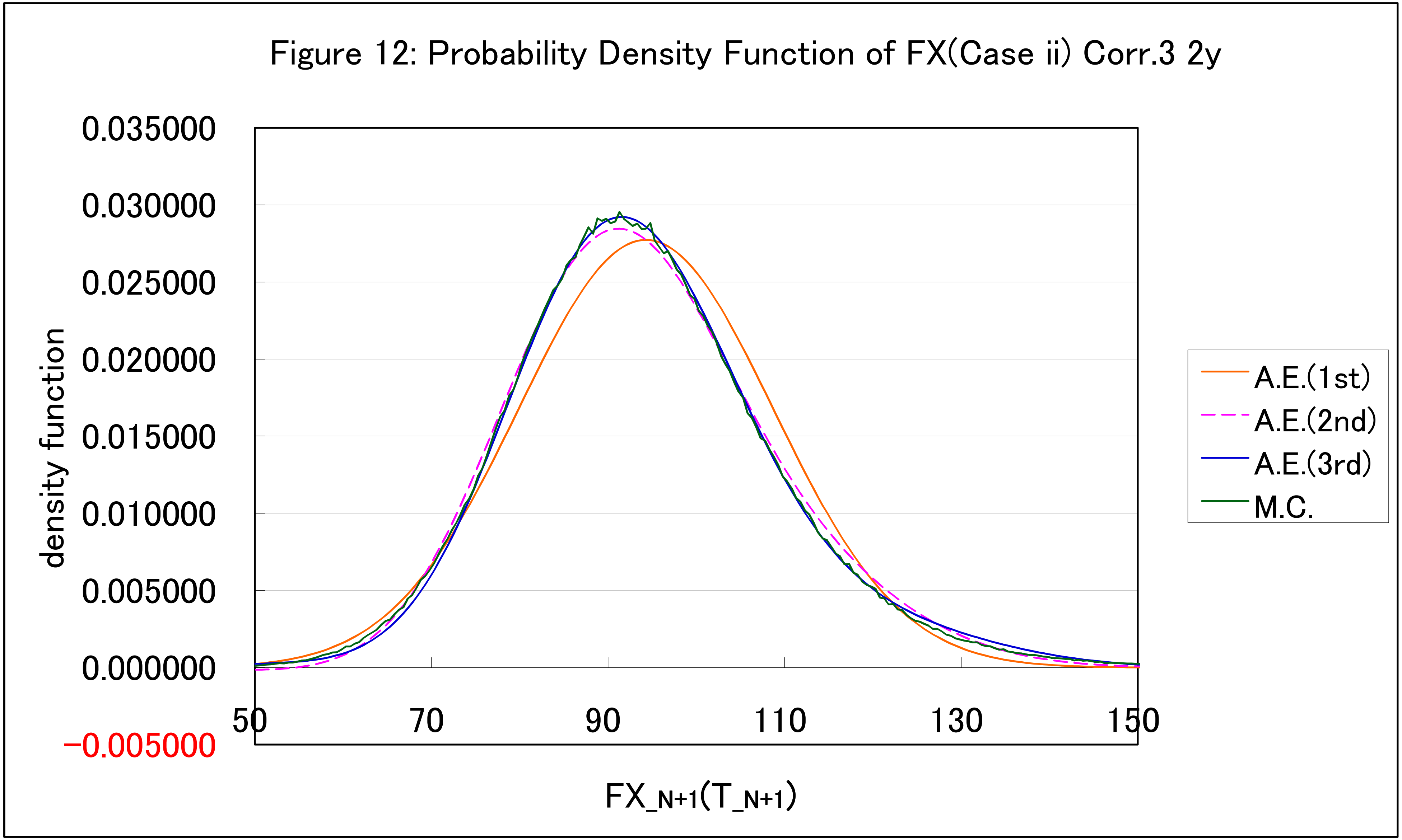

Figure 12: Probability Density Function of FX(Case ii) Corr.3 2y
0.035000
0.030000
0.025000
0.020000
0.015000
0.010000
0.005000
0.000000
−0.005000
density function
50
70
90
110
130
150
FX_N+1(T_N+1)
A.E.(1st)
A.E.(2nd)
A.E.(3rd)
M.C.

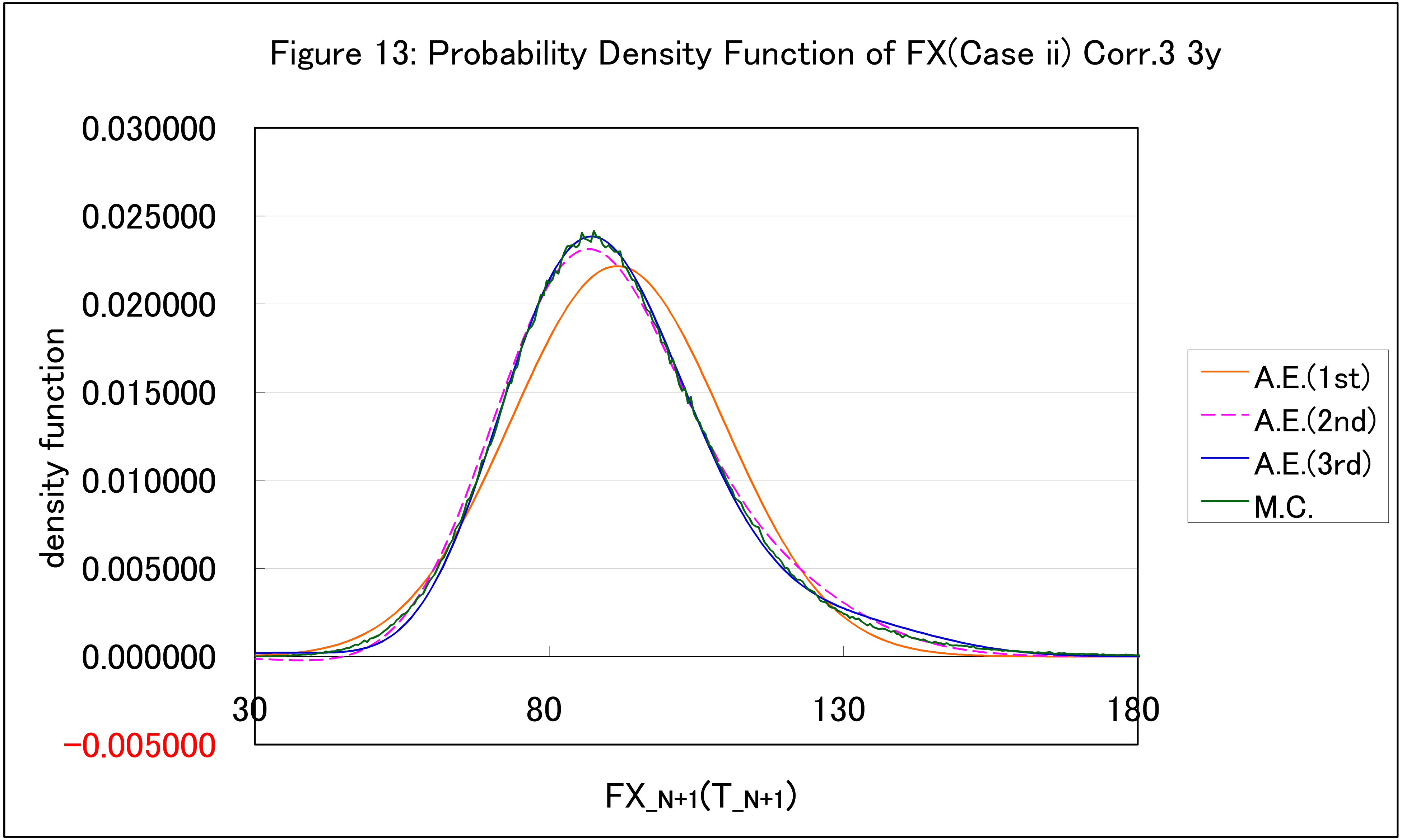
Figure 13: Probability Density Function of FX(Case ii) Corr.3 3y
density function
0.030000
0.025000
0.020000
0.015000
0.010000
0.005000
0.000000
−0.005000
30
80
130
180
FX_N+1(T_N+1)
A.E.(1st)
A.E.(2nd)
A.E.(3rd)
M.C.

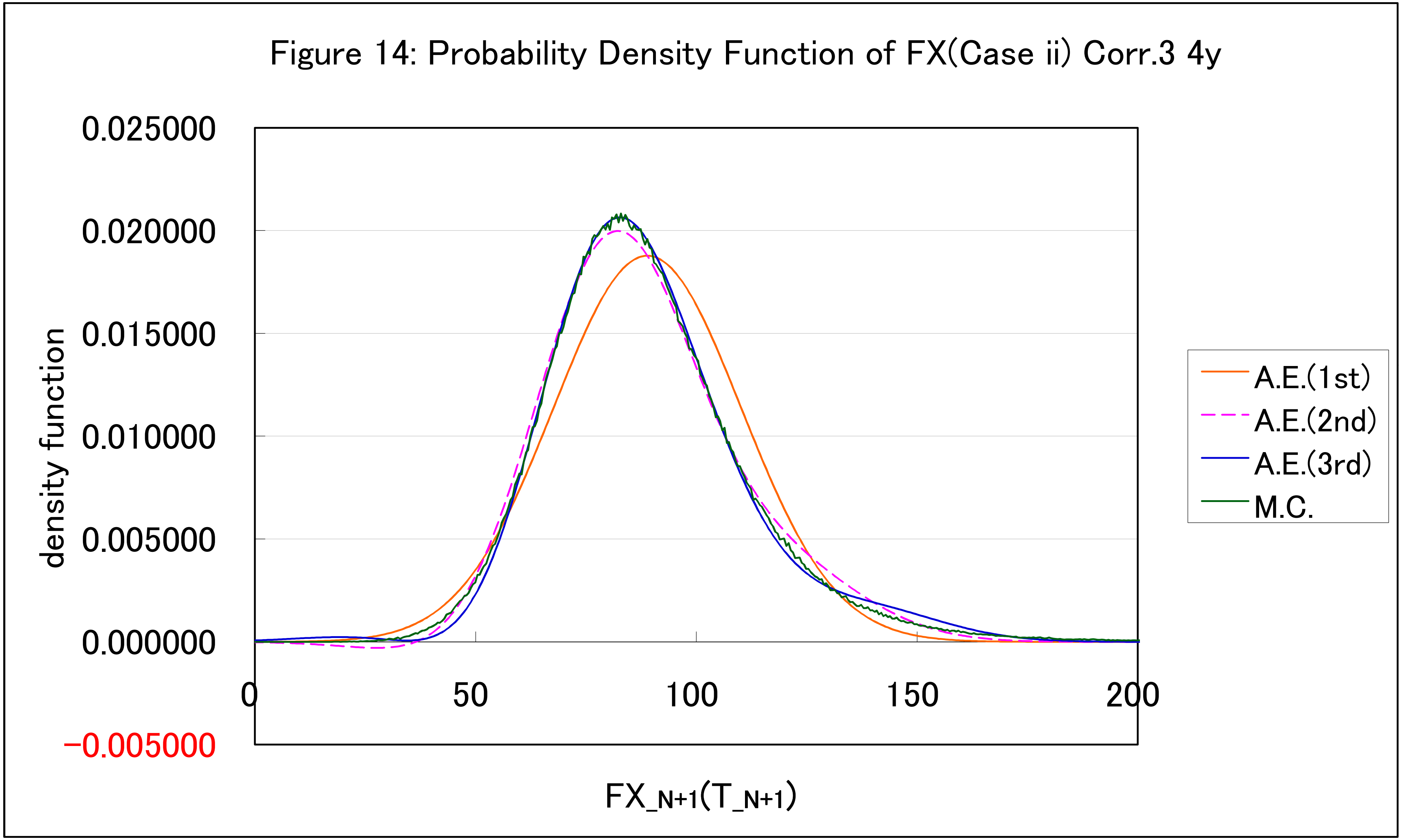
Figure 14: Probability Density Function of FX(Case ii) Corr.3 4y
density function
0.025000
0.020000
0.015000
0.010000
0.005000
0.000000
−0.005000
0
50
100
150
200
FX_N+1(T_N+1)
A.E.(1st)
A.E.(2nd)
A.E.(3rd)
M.C.

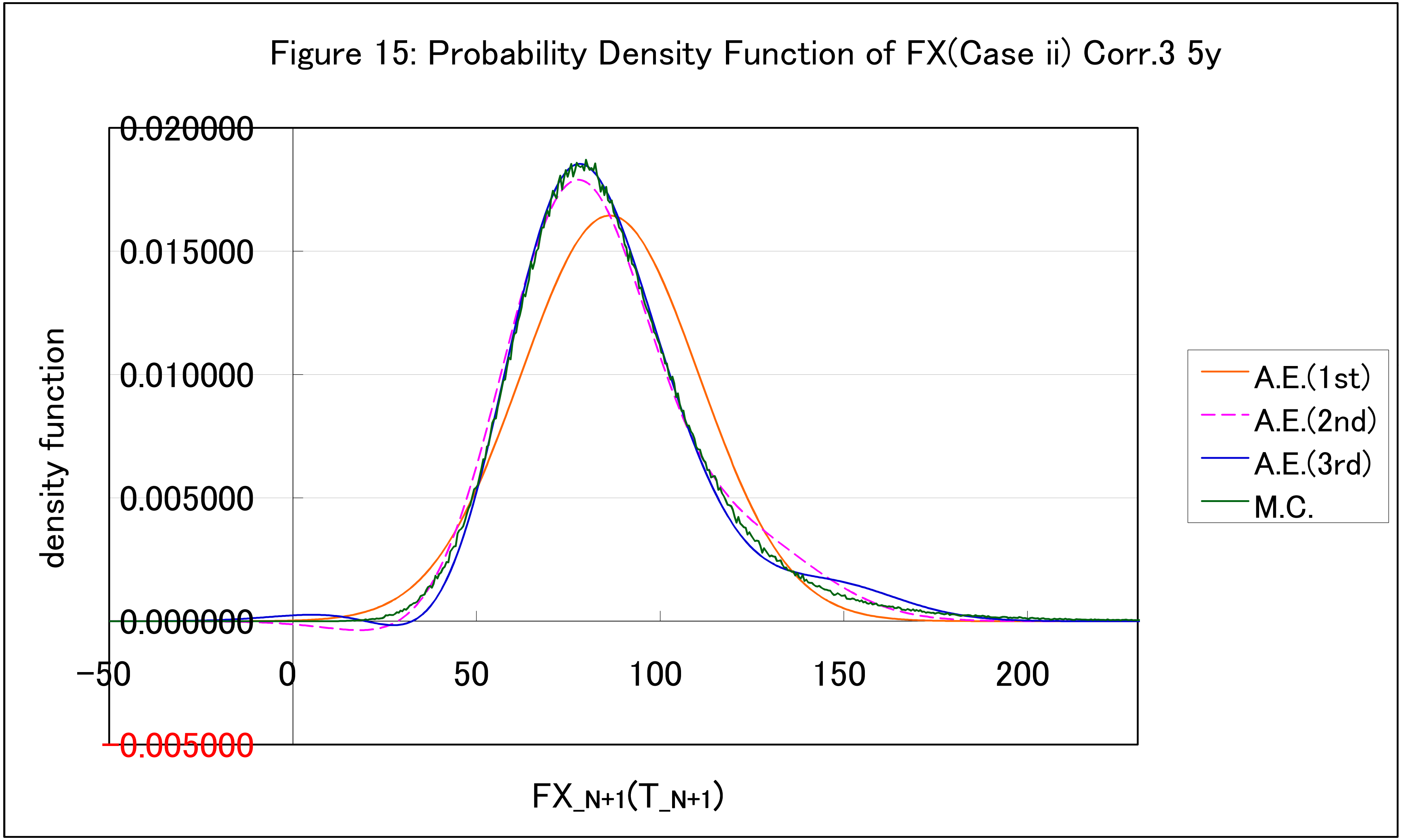
Figure 15: Probability Density Function of FX(Case ii) Corr.3 5y
density function
0.020000
0.015000
0.010000
0.005000
0.000000
-0.005000
-50
0
50
100
150
200
FX_N+1(T_N+1)
A.E.(1st)
A.E.(2nd)
A.E.(3rd)
M.C.

Figure 16: Absolute Values of Differences:Corr.1 ITM

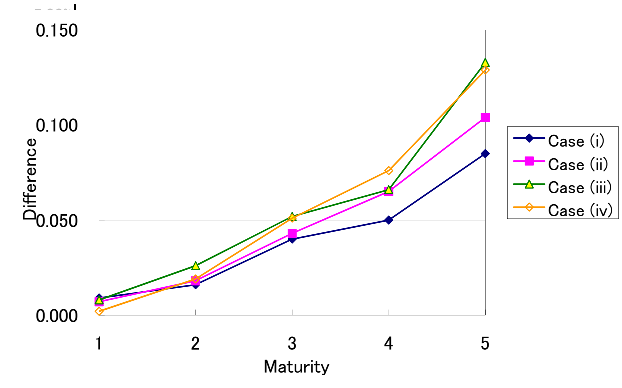

Figure 17: Absolute Values of Relative Differences:Corr.1 ITM

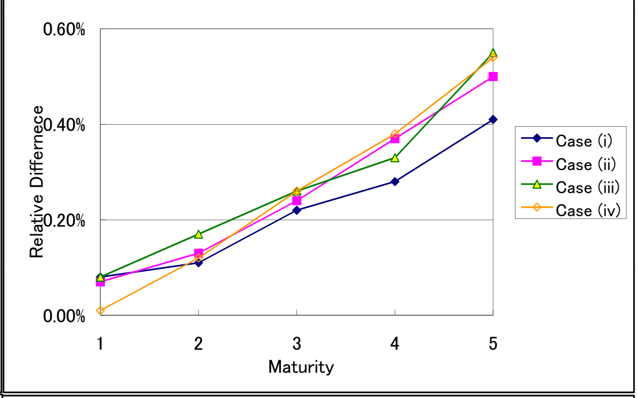

Figure 18: Absolute Values of Differences:Corr.2 ITM

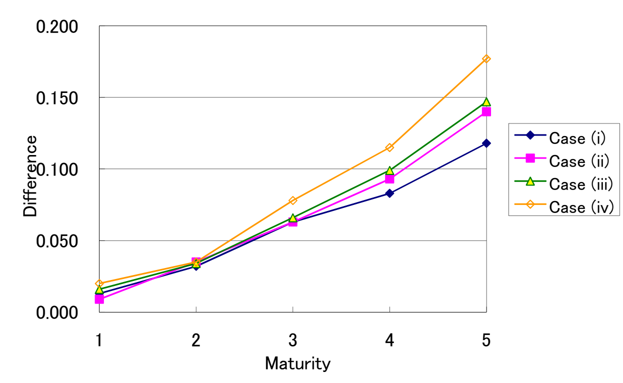

Figure 19: Absolute Values of Relative Differences:Corr.2 ITM

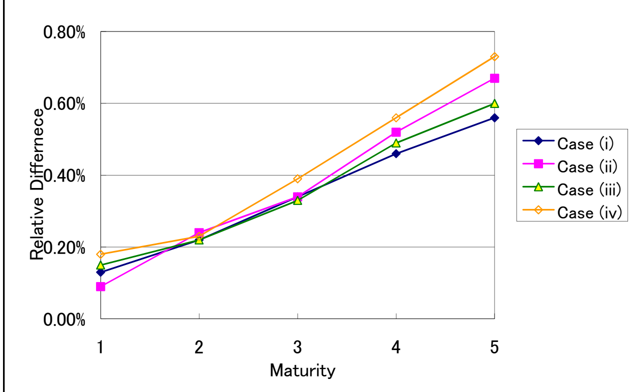

Figure 20: Absolute Values of Differences:Corr.3 ITM

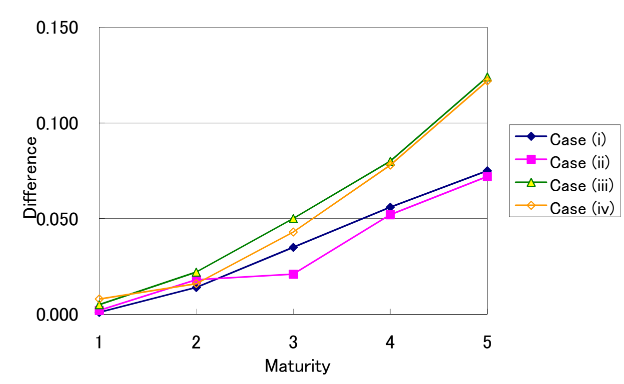

Figure 21: Absolute Values of Relative Differences:Corr.3 ITM

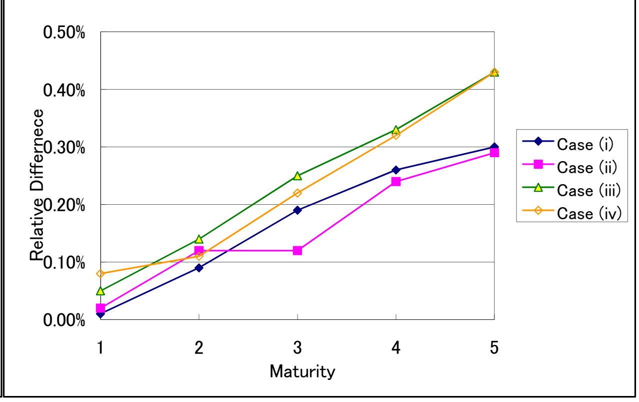

Figure 22: Absolute Values of Differences:Corr.1 ATM

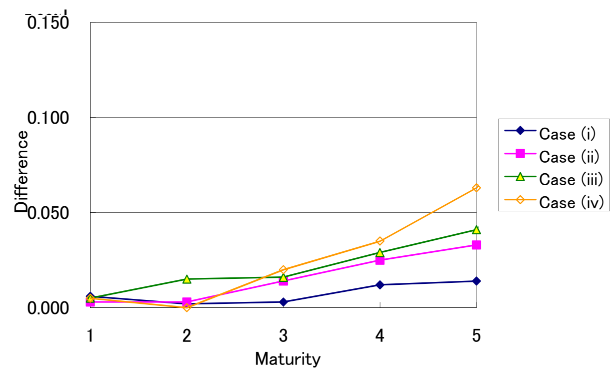

Figure 23: Absolute Values of Relative Differences:Corr.1 ATM

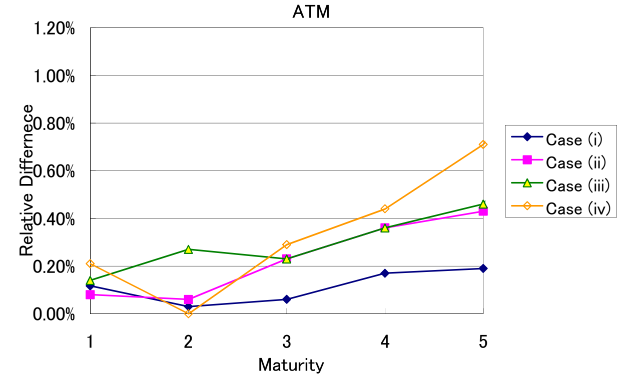

Figure 24: Absolute Values of Differences:Corr.2 ATM

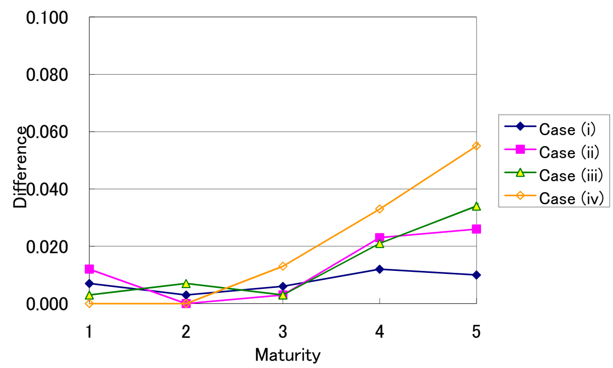

Figure 25: Absolute Values of Relative Differences:Corr.2 ATM

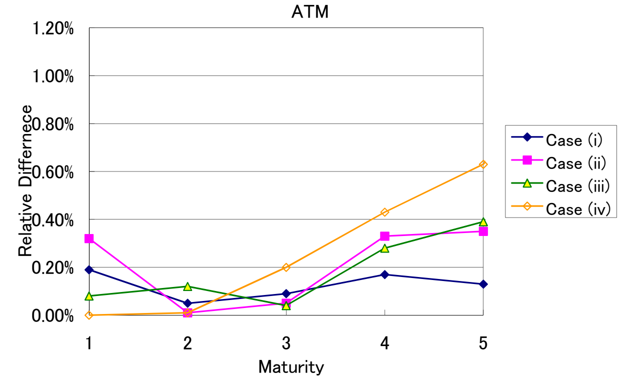

Figure 26: Absolute Values of Differences:Corr.3 ATM

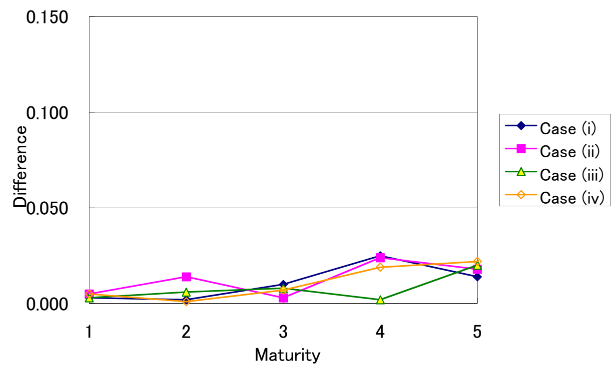

Figure 27: Absolute Values of Relative Differences:Corr.3 ATM

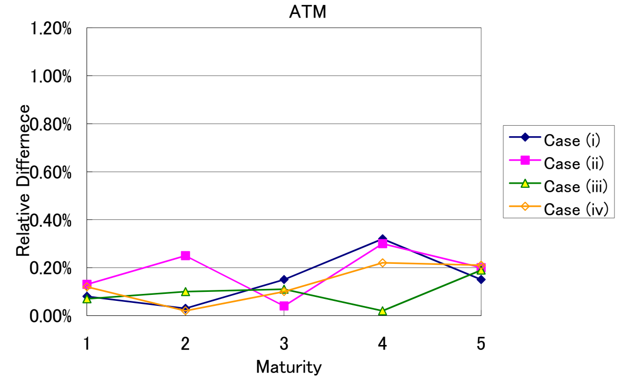

Figure 28: Absolute Values of Differences:Corr.1 OTM

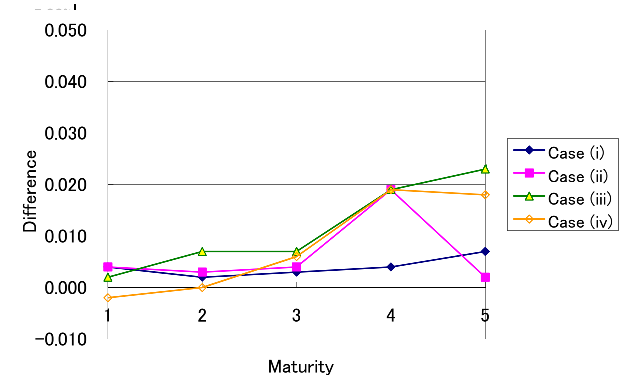

Figure 29: Absolute Values of Relative Differences:Corr.1 OTM

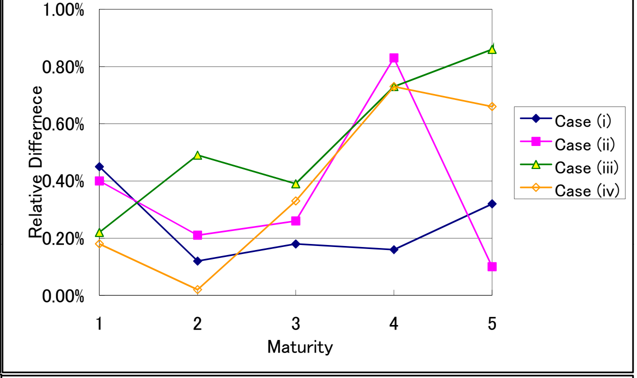

Figure 30: Absolute Values of Differences:Corr.2 OTM

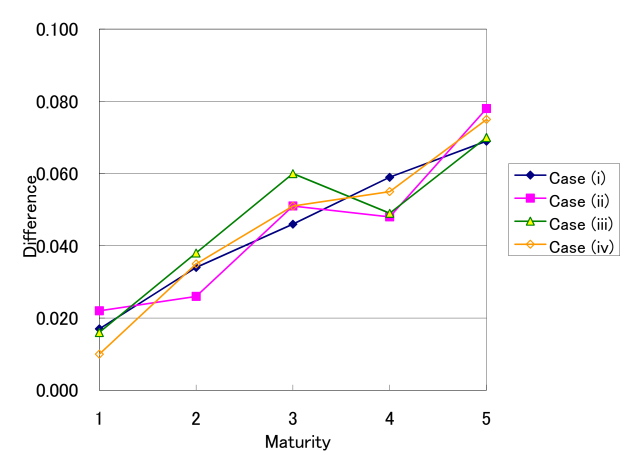

Figure 31: Absolute Values of Relative Differences:Corr.2 OTM

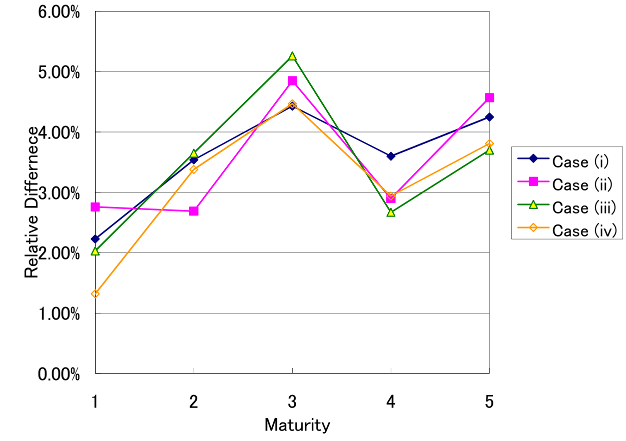

Figure 32: Absolute Values of Differences:Corr.3 OTM

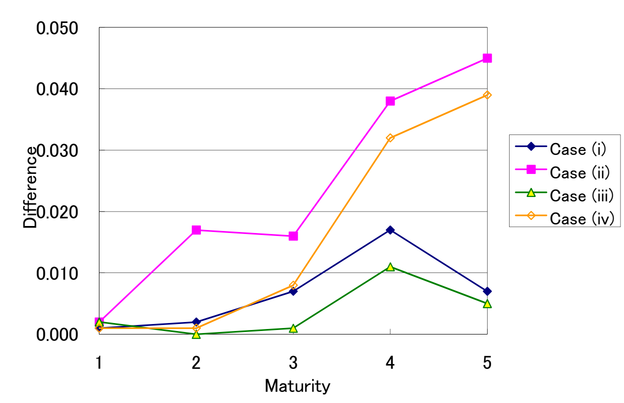

Figure 33: Absolute Values of Relative Differences:Corr.3 OTM

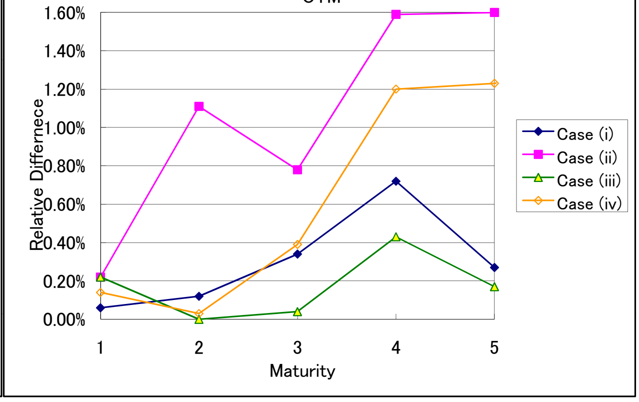

Table 6: Calibrated parameters and Implied volatilities on Jun 26 2006.

Date 26-Jun-06 S(0) 116.40

Calibration(1y,2y,3y,5y)

Parameters

$\sigma(0)=\theta$	8.3%
κ	1.113
ω^*	0.294

Correlations

	Domestic Int	Foreign Int.	Spot Forex	Spot Forex's Vol.
Domestic Int.	1	0.477	-0.776	-0.413
Foreign Int.		1	-0.222	0.851
Spot Forex			1	-0.325
Spot Forex's Vol.				1

Market Implied Volatilities

Delta	10	25	50	25	10
1y	11.51%	9.85%	8.73%	8.20%	8.13%
2y	12.15%	10.09%	8.73%	8.01%	8.00%
3y	12.30%	10.26%	8.73%	7.91%	7.84%
5y	12.85%	10.83%	8.88%	7.78%	7.36%

Model Implied Volatilities

Delta	10	25	50	25	10
1y	11.39%	9.79%	8.75%	8.20%	8.20%
2y	12.20%	10.20%	8.72%	7.95%	8.04%
3y	12.43%	10.38%	8.80%	7.83%	7.77%
5y	12.71%	10.70%	9.00%	7.72%	7.40%

Differences

Delta	10	25	50	25	10
1y	-0.12%	-0.06%	0.03%	0.00%	0.07%
2y	0.05%	0.11%	-0.01%	-0.07%	0.03%
3y	0.13%	0.12%	0.07%	-0.08%	-0.06%
5y	-0.14%	-0.12%	0.13%	-0.06%	0.04%

Figure 34: Market and Model Smiles on Jun 26 2006.

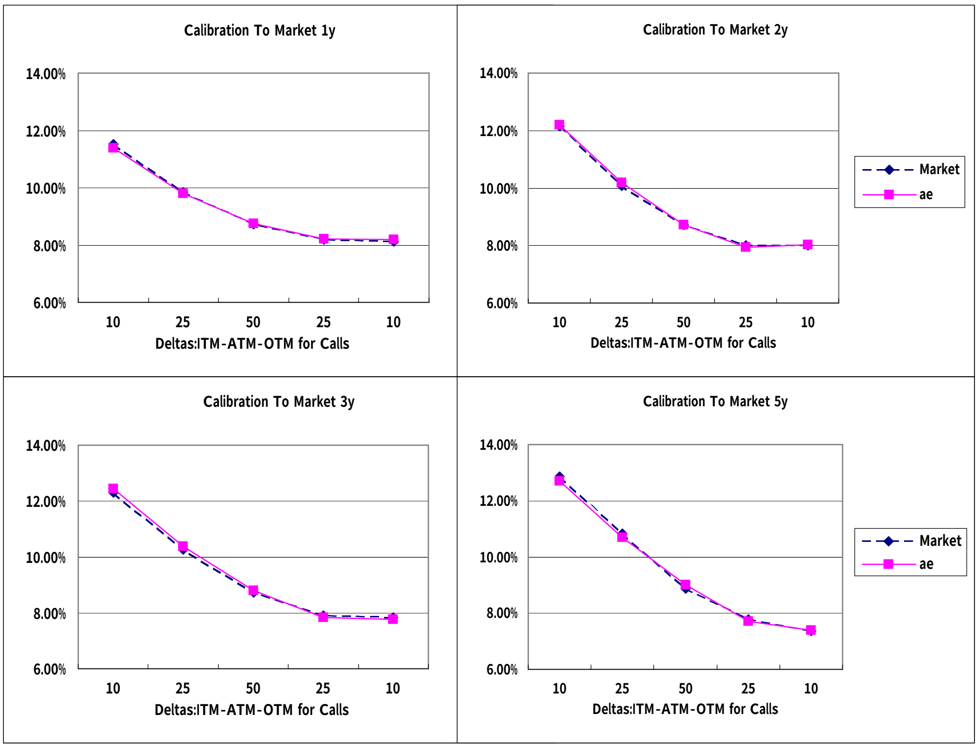

Figure 35: Market surface and Model surface on Jun 26 2006.

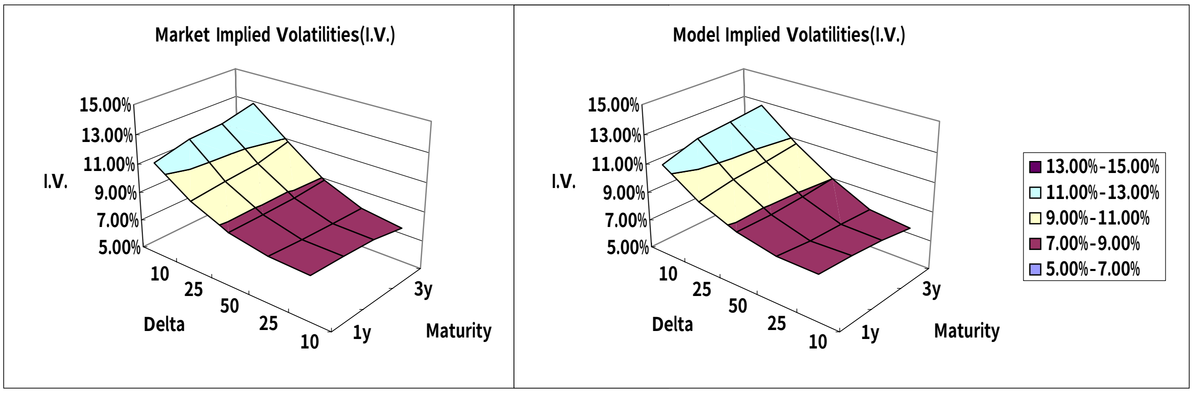

Table 7: Calibrated parameters and Implied volatilities on Jul 05 2006.

Date 5-Jul-06 S(0) 115.17

Calibration(1y,2y,3y,5y)

Parameters

$\sigma(0)=\theta$	8.4%
κ	1.054
ω^*	0.286

Correlations

	Domestic Int	Foreign Int.	Spot Forex	Spot Forex's Vol.
Domestic Int.	1	-0.123	-0.888	-0.443
Foreign Int.		1	-0.194	0.816
Spot Forex			1	-0.372
Spot Forex's Vol.				1

Market Implied Volatilities

Delta	10	25	50	25	10
1y	11.55%	9.94%	8.78%	8.19%	8.05%
2y	12.20%	10.08%	8.70%	7.98%	8.00%
3y	12.37%	10.23%	8.70%	7.88%	7.91%
5y	13.11%	10.95%	8.90%	7.75%	7.35%

Model Implied Volatilities

Delta	10	25	50	25	10
1y	11.42%	9.79%	8.74%	8.14%	8.08%
2y	12.29%	10.13%	8.63%	7.85%	7.98%
3y	12.56%	10.33%	8.71%	7.73%	7.75%
5y	12.99%	10.81%	9.06%	7.75%	7.43%

Differences

Delta	10	25	50	25	10
1y	-0.13%	-0.15%	-0.04%	-0.05%	0.03%
2y	0.10%	0.06%	-0.07%	-0.13%	-0.02%
3y	0.19%	0.10%	0.01%	-0.15%	-0.16%
5y	-0.13%	-0.13%	0.16%	0.01%	0.08%

Figure 36: Market and Model Smiles on Jul 05 2006.

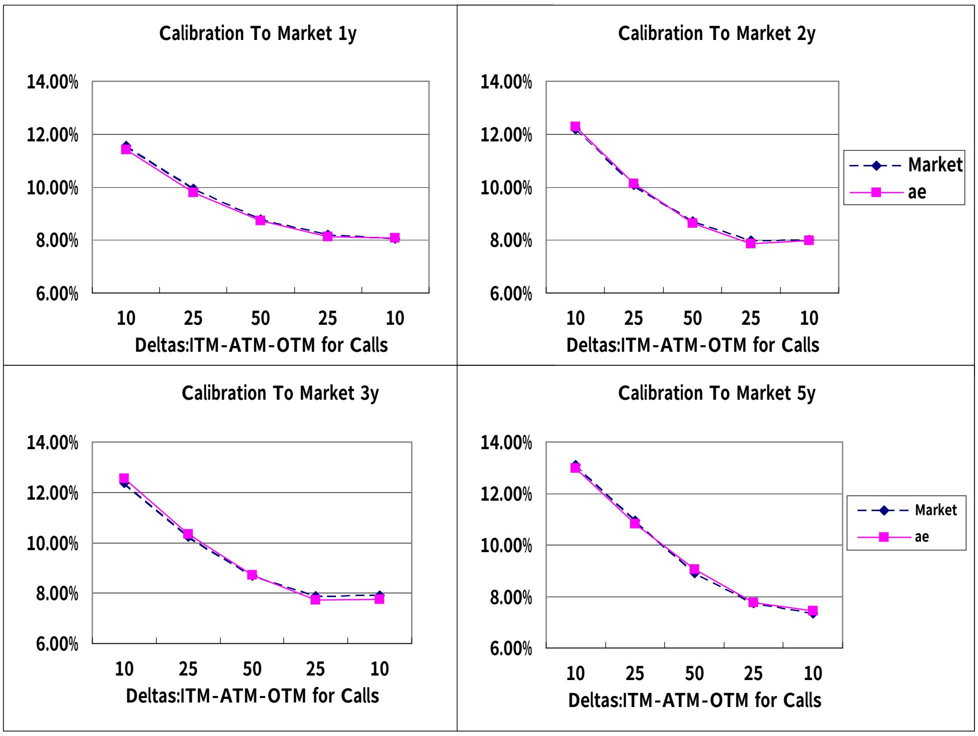

Figure 37: Market surface and Model surface on Jul 05 2006.